Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 49(2)(2017) pp. 127-151

New Refinement of Jensen-Mercer's Operator Inequality and Applications to Means

Khuram Ali Khan
Department of Mathematics,
University of Sargodha, Pakistan.
Email: khuramsms@gmail.com

Muhammad Adil Khan
Department of Mathematics,
University of Peshawer, Pakistan.
Email: adilswati@gmail.com

Uzma Sadaf
Department of Mathematics,
University of Sargodha, Pakistan.
Email: uzmasadaf06@gmail.com

Received: 11 July, 2016 / Accepted: 02 June, 2017 / Published online: 08 June, 2017

Abstract. In this paper we give the refinement of Jensen-Mercer's and power mean inequalities for operators. We launch the corresponding mixed symmetric means for strictly positive operators defined on Hilbert space and also establish the refinement of inequalities of power means for these operators.

AMS (MOS) Subject Classification Codes: 26D07, 46E63, 46E64
Key Words: Operator convex functions, Mixed symmetric means, Power mean inequality, Jensen-Mercer's inequality.

1. Introduction and Preliminary Results

Let H be a complex Hilbert space and $S(I)$ be the class of all self-adjoint bounded operators on H whose spectra are contained in an interval $I \subset \mathbb{R}$. The spectrum of a bounded operator A on H is denoted by $\text{sp}(A)$. Let $I \subset \mathbb{R}$ be an interval then the function $f : S(I) \to \mathbb{R}$ is said to be operator-convex if f is continuous on $S(I)$ and

$$f(sA + tB) \leq sf(A) + tf(B)$$

for all $A, B \in S(I)$ and for all positive numbers s and t such that $s + t = 1$. The function f is called operator-concave on $S(I)$ if $-f$ is operator-convex on $S(I)$.

Theorem 1.1. *Jensen's operator inequality ([1]): Let $I \subset \mathbb{R}$ be an interval and $f : S(I) \to \mathbb{R}$ be an operator-convex function on I. If $A_i \in S(I)$ and $w_i > 0$; $i = 1, 2, ..., n$ such that $W_n = \sum_{i=1}^{n} w_i$, then*

$$f(\frac{1}{W_n}\sum_{i=1}^{n} w_i A_i) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(A_i). \tag{1. 1}$$

If f is an operator-concave function, then (1. 1) is reversed.

A self-adjoint bounded operator A on H is called strictly positive if it is positive and invertible, or equivalently, $\mathrm{Sp}(A) \subset [m, M]$ for some $0 < m < M$. The power mean for strictly positive operators $\mathbf{A} := (A_1, ..., A_n)$ with positive weights $\mathbf{w} := (w_1, ..., w_n)$ is defined in [1] as

$$M_r(\mathbf{w}, \mathbf{A}) := (\frac{1}{W_n}\sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}},$$

where $r \in \mathbb{R} \setminus \{0\}$ and $W_n = \sum_{i=1}^{n} w_i$.

The next result is borrowed from [1] (see also [2]).

Theorem 1.2. *Let $I \subset \mathbb{R}$ be an interval and $\mathbf{A}$ be an n-tuple of strictly positive operators with positive weights $\mathbf{w} := (w_1, ..., w_n)$ such that $W_n = \sum_{i=1}^{n} w_i$. Then*

$$M_s(\boldsymbol{w}, \boldsymbol{A}) \leq M_r(\boldsymbol{w}, \boldsymbol{A}); \quad r, s \in \mathbb{R} \setminus \{0\}$$

if either

(i) $s \leq r; r, s \notin (-1, 1)$ *or*
(ii) $\frac{1}{2} \leq s \leq 1 \leq r$ *or*
(iii) $s \leq -1 \leq r \leq -\frac{1}{2}$ *holds.*

The inequalities for convex function are widely studied e.g see [8, 9, 11, 12, 13] and references with in. Jensen-Mercer operator inequality and refinements of the operator Jensen-Mercer inequality are studied in [6] and [7] respectively, where as a variant of the Jensen-Mercer operator inequality for superquadratic functions is discussed in [4]. We give a new refinement of Jensen-Mercer operator inequality for operator-convex in the next section.

2. Main Results

Let $\phi : I \to \mathbb{R}$ be an operator convex function on $I \subset \mathbb{R}$. If $A_i \in S(I)$ and $w_i > 0$ with $W_n = \sum_{i=1}^{n} w_i = 1$ then for any nonempty proper subset J of $\{1, 2, ..., n\}$ we put $\bar{J} := \{1, 2, ..., n\} \setminus J$ and define $W_J = \sum_{i \in J} w_i$ and $W_{\bar{J}} = 1 - \sum_{i \in J} w_i$. Now for an operator convex function ϕ, the n-tuples $\mathbf{A} := (A_1, ..., A_n)$ and $\mathbf{w} := (w_1, ..., w_n)$ as above, we can define the following functional

$$D(\phi, \mathbf{w}, \mathbf{A}; J) := W_J \phi(\frac{1}{W_J}\sum_{i \in J} w_i A_i) + W_{\bar{J}} \phi(\frac{1}{W_{\bar{J}}}\sum_{i \in \bar{J}} w_i A_i).$$

In case if $J = \{k\}$, $k \in \{1, 2, ..., n\}$ then we have functional

$$D_k(\phi, \mathbf{w}, \mathbf{A}) := D(\phi, \mathbf{w}, \mathbf{A}; \{k\}) = w_k\phi(A_k) + (1 - w_k)\phi\left(\frac{\sum\limits_{i=1}^{n} w_iA_i - w_kA_k}{1 - w_k}\right),$$

which has been investigated for convex function in [5] and earlier in [3].

Theorem 2.1. *Let $\phi : S(I) \to \mathbb{R}$ be an operator convex function on $I \subset \mathbb{R}$. If $A_i \in S(I)$ and $w_i > 0$; $i = 1, 2, ..., n$ with $W_n = \sum\limits_{i=1}^{n} w_i = 1$ then for any nonempty proper subset J of $\{1, 2, ..., n\}$,*

$$\sum_{i=1}^{n} w_i\phi(A_i) \geq D(\phi, \mathbf{w}, \mathbf{A}; J) \geq \phi(\sum_{i=1}^{n} w_iA_i). \tag{2. 2}$$

If ϕ is concave, then (2. 2) *is reversed.*

Proof. By the convexity of the function ϕ, we have

$$\begin{aligned} D(\phi, \mathbf{w}, \mathbf{A}; J) &= W_J\phi(\frac{1}{W_J}\sum_{i\in J} w_iA_i) + W_{\bar{J}}\phi(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_iA_i) \\ &\geq \phi(W_J(\frac{1}{W_J}\sum_{i\in J} w_iA_i) + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_iA_i)) \\ &= \phi(\sum_{i=1}^{n} w_iA_i), \end{aligned}$$

which implies that

$$D(\phi, \mathbf{w}, \mathbf{A}; J) \geq \phi(\sum_{i=1}^{n} w_iA_i). \tag{2. 3}$$

Now

$$\begin{aligned} \sum_{i=1}^{n} w_i\phi(A_i) &= \sum_{i\in J} w_i\phi(A_i) + \sum_{i\in \bar{J}} w_i\phi(A_i \\ &= W_J(\frac{1}{W_J}\sum_{i\in J} w_i\phi(A_i)) + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i\phi(A_i)) \\ &\geq W_J\phi(\frac{1}{W_J}\sum_{i\in J} w_iA_i) + W_{\bar{J}}\phi(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_iA_i) \\ &= D(\phi, \mathbf{w}, \mathbf{A}; J), \end{aligned}$$

which implies that

$$\sum_{i=1}^{n} w_i\phi(A_i) \geq D(\phi, \mathbf{w}, \mathbf{A}; J). \tag{2. 4}$$

Hence by combining (2. 3) and (2. 4), we get (2. 2). □

Now for fixed values of k we assume the (non-empty) subsets $J_i^k, ..., J_l^k$ of $\{1, 2, ..., n\}$, $1 \leq l \leq k \leq n$ such that $J_i^k \cup ... \cup J_l^k = \{1, 2, ..., n\}$ and $J_i^k \cap J_j^k = \phi$ for $i \neq j$, where $i, j = 1, 2, ..., n$. Consider the functional

$$D_k\overset{J_1^k,...,J_l^k}{(\phi, \mathbf{w}, \mathbf{A})} := \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} \phi((a+b)I - \frac{1}{W_{J_j^k}} \sum_{i \in J_l^k} w_i A_i),$$

for the function $\phi : S([a, b]) \to \mathbb{R}$ where **A** is an n-tuple of selfadjoint operators defined on $[a, b] \subset \mathbb{R}$ and $w_i > 0$; $i = 1, 2, ..., n$ also $W_J = \sum\limits_{i \in J} w_i$; $J \subset \{1, 2, ..., n\}$.

For convex functions see [10].

Theorem 2.2. *Let* **A** *be an n-tuple of selfadjoint operators defined on $[a, b] \subset \mathbb{R}$ with $w_i > 0$; $i = 1, 2, ..., n$ such that $W_n = \sum\limits_{i=1}^{n} w_i$. If $\phi : S([a, b]) \to \mathbb{R}$ is an operator-convex function, then*

$$\begin{aligned} &\phi(aI) + \phi(bI) - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i \phi(A_i) \geq M_k \geq M_{k-1} \geq ... \geq M_2 \geq \\ &\geq \phi((a+b)I - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i A_i), \end{aligned} \tag{2. 5}$$

where

$$M_k := \max_{J_1^k,...,J_l^k} [D_k\overset{J_1^k,...,J_l^k}{(\phi, \boldsymbol{w}, \boldsymbol{A})}]$$

If f is an operator-concave function, then (2. 5) *is reversed.*

Proof. First we show that

$$M_m \geq M_{m-1}; \quad m = 3, ..., k,$$

Since

$$D_{m-1}\overset{J_1^{m-1},...,J_{l-1}^{m-1}}{(\phi, \mathbf{w}, \mathbf{A})} := \sum_{j=1}^{l-1} \frac{W_{J_j^{m-1}}}{W_n} \phi((a+b)I - \frac{1}{W_{J_j^{m-1}}} \sum_{i \in J_l^{m-1}} w_i A_i).$$

So let us assume that the maximum of the functional $D_{m-1}\overset{J_1^{m-1},...,J_{l-1}^{m-1}}{(\phi, \mathbf{w}, \mathbf{A})}$ is obtained for $J_1^{m-1} = J_{g_1}^{k-1}, J_2^{m-1} = J_{g_2}^{m-1}, ..., J_{l-1}^{m-1} = J_{g_{l-1}}^{m-1}$. i.e

$$\max_{J_1^{m-1},...,J_{l-1}^{m-1}} [D_{m-1}\overset{J_1^{m-1},...,J_{l-1}^{m-1}}{(\phi, \mathbf{w}, \mathbf{A})}] = \sum_{j=g_1}^{l-1} \frac{W_{J_j^{m-1}}}{W_n} \phi((a+b)I - \frac{1}{W_{J_j^{m-1}}} \sum_{i \in J_l^{m-1}} w_i A_i)$$

$$= \frac{W_{J_{g_1}^{m-1}}}{W_n} \phi((a+b)I - \frac{1}{W_{J_{g_1}^{m-1}}} \sum_{i \in J_{g_1}^{m-1}} w_i A_i) + \frac{W_{J_{g_2}^{m-1}}}{W_n} \phi((a+b)I - \frac{1}{W_{J_{g_2}^{m-1}}} \sum_{i \in J_{g_2}^{m-1}} w_i A_i) +$$

$$... + \frac{W_{J_{g_{l-1}}^{m-1}}}{W_n} \phi((a+b)I - \frac{1}{W_{J_{g_{l-2}}^{m-1}}} \sum_{i \in J_{g_{l-2}}^{m-1}} w_i A_i).$$

Without loss of generality, we may assume that $J_{g_{l-1}}^{m-1}$ contains more than one point, so we choose two non-empty sets $J_{r_1}^{m-1}$ and $J_{r_2}^{m-1}$ such that $J_{r_1}^{m-1} \cap J_{r_2}^{m-1} = \phi$ and $J_{r_1}^{m-1} \cup J_{r_2}^{m-1} = J_{g_{l-1}}^{m-1}$. Then

$$\frac{W_{J_{g_1}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_1}^{m-1}}}\sum_{i\in J_{g_1}^{m-1}} w_iA_i)+$$
$$+...+\frac{W_{J_{g_{l-2}}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_{l-2}}^{m-1}}}\sum_{i\in J_{g_{l-2}}^{m-1}} w_iA_i)$$
$$+\frac{W_{J_{g_{l-1}}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_{l-1}}^{m-1}}}\sum_{i\in J_{g_{l-1}}^{m-1}} w_iA_i)$$

$$= \frac{W_{J_{g_1}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_1}^{m-1}}}\sum_{i\in J_{g_1}^{m-1}} w_iA_i)+$$
$$+...+\frac{W_{J_{g_{l-2}}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_{l-2}}^{m-1}}}\sum_{i\in J_{g_{l-2}}^{m-1}} w_iA_i)$$
$$+\frac{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}\sum_{i\in J_{r_1}^{m-1}\cup J_{r_2}^{m-1}} w_iA_i)$$

$$= \frac{W_{J_{g_1}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_1}^{m-1}}}\sum_{i\in J_{g_1}^{m-1}} w_iA_i)+$$
$$+...+\frac{W_{J_{g_{l-2}}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_{l-2}}^{m-1}}}\sum_{i\in J_{g_{l-2}}^{m-1}} w_iA_i)$$
$$+\frac{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}{W_n}\phi(\frac{1}{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}\sum_{i\in J_{r_1}^{m-1}\cup J_{r_2}^{m-1}} w_i((a+b)I - A_i))$$

$$= \frac{W_{J_{g_1}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_1}^{m-1}}}\sum_{i\in J_{g_1}^{m-1}} w_iA_i)+$$
$$+...+\frac{W_{J_{g_{l-2}}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_{l-2}}^{m-1}}}\sum_{i\in J_{g_{l-2}}^{m-1}} w_iA_i)$$
$$+\frac{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}{W_n}\phi(\frac{1}{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}\sum_{i\in J_{r_1}^{m-1}} w_i((a+b)I - A_i)$$
$$+\frac{1}{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}\sum_{i\in J_{r_2}^{m-1}w_i}((a+b)I - A_i))$$

$$= \frac{W_{J_{g_1}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_1}^{m-1}}}\sum_{i\in J_{g_1}^{m-1}} w_iA_i)+$$
$$+...+\frac{W_{J_{g_{l-2}}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_{l-2}}^{m-1}}}\sum_{i\in J_{g_{l-2}}^{m-1}} w_iA_i)$$
$$+\frac{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}{W_n}\phi(\frac{W_{J_{r_1}^{m-1}}}{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}\frac{1}{W_{J_{r_1}^{m-1}}}\sum_{i\in J_{r_1}^{m-1}} w_i((a+b)I - A_i)$$
$$+\frac{W_{J_{r_2}^{m-1}}}{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}\frac{1}{W_{J_{r_2}^{m-1}}}\sum_{i\in J_{r_2}^{m-1}} w_i((a+b)I - A_i))$$

$$= \frac{W_{J_{g_1}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_1}^{m-1}}}\sum_{i\in J_{g_1}^{m-1}} w_iA_i) +$$
$$+ \ldots + \frac{W_{J_{g_{l-2}}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_{l-2}}^{m-1}}}\sum_{i\in J_{g_{l-2}}^{m-1}} w_iA_i)$$
$$+ \frac{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}{W_n}\phi(\frac{W_{J_{r_1}^{m-1}}}{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}((a+b)I - \frac{1}{W_{J_{r_1}^{m-1}}}\sum_{i\in J_{r_1}^{m-1}} w_iA_i) +$$
$$\frac{W_{J_{r_2}^{m-1}}}{W_{J_{r_1}^{m-1}\cup J_{r_2}^{m-1}}}((a+b)I - \frac{1}{W_{J_{r_2}^{m-1}}}\sum_{i\in J_{r_2}^{m-1}} w_iA_i))$$

$$\leq \frac{W_{J_{g_1}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_1}^{m-1}}}\sum_{i\in J_{g_1}^{m-1}} w_iA_i) +$$
$$+ \ldots + \frac{W_{J_{g_{l-2}}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_{l-2}}^{m-1}}}\sum_{i\in J_{g_{l-2}}^{m-1}} w_iA_i)$$
$$+ \frac{W_{J_{r_1}}^{m-1}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{r_1}^{m-1}}}\sum_{i\in J_{r_1}^{m-1}} w_iA_i)$$
$$+ \frac{W_{J_{r_2}}^{m-1}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{r_2}^{m-1}}}\sum_{i\in J_{r_2}^{m-1}} w_iA_i),$$

Hence

$$\max_{J_1^{m-1},\ldots,J_{l-1}^{m-1}} [D_{m-1}^{J_1^{m-1},\ldots,J_{l-1}^{m-1}}(\phi, \mathbf{w}, \mathbf{A})]$$
$$\leq \frac{W_{J_{g_1}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_1}^{m-1}}}\sum_{i\in J_{g_1}^{m-1}} w_iA_i)$$
$$+$$
$$\vdots$$
$$+ \frac{W_{J_{g_{l-2}}^{m-1}}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{g_{l-2}}^{m-1}}}\sum_{i\in J_{g_{l-2}}^{m-1}} w_iA_i)$$
$$+ \frac{W_{J_{r_1}}^{m-1}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{r_1}^{m-1}}}\sum_{i\in J_{r_1}^{m-1}} w_iA_i)$$
$$+ \frac{W_{J_{r_2}}^{m-1}}{W_n}\phi((a+b)I - \frac{1}{W_{J_{r_2}^{m-1}}}\sum_{i\in J_{r_2}^{m-1}} w_iA_i)$$

But

$$\max_{J_1^m,\dots,J_l^m}[D_m\overset{J_1^m,\dots,J_l^m}{(\phi,\mathbf{w},\mathbf{A})}]$$
$$\geq \frac{W_{J_{g_1}^{m-1}}}{W_n}\phi((a+b)I-\tfrac{1}{W_{J_{g_1}^{m-1}}}\sum_{i\in J_{g_1}^{m-1}}w_iA_i)$$
$$+$$
$$\vdots$$
$$+\frac{W_{J_{g_{l-2}}^{m-1}}}{W_n}\phi((a+b)I-\tfrac{1}{W_{J_{g_{l-2}}^{m-1}}}\sum_{i\in J_{g_{l-2}}^{m-1}}w_iA_i)$$
$$+\frac{W_{J_{r_1}}^{m-1}}{W_n}\phi((a+b)I-\tfrac{1}{W_{J_{r_1}^{m-1}}}\sum_{i\in J_{r_1}^{m-1}}w_iA_i)$$
$$+\frac{W_{J_{r_2}}^{m-1}}{W_n}\phi((a+b)I-\tfrac{1}{W_{J_{r_2}^{m-1}}}\sum_{i\in J_{r_2}^{m-1}}w_iA_i).$$

From above we have

$$\max_{J_1^m,\dots,J_l^m}[D_k\overset{J_1^m,\dots,J_l^m}{(\phi,\mathbf{w},\mathbf{A})}]\geq \max_{J_1^{m-1},\dots,J_{l-1}^{m-1}}[D_{m-1}\overset{J_1^{m-1},\dots,J_{l-1}^{m-1}}{(\phi,\mathbf{w},\mathbf{A})}],$$

i.e

$$M_m\geq M_{m-1};\quad m=3,\dots,k.$$

Next we have to show that

$$\phi(aI)+\phi(bI)-\frac{1}{W_n}\sum_{i=1}^n w_i\phi(A_i)\geq M_k.$$

For this we assume that maximum of the functional $D_k\overset{J_1^k,\dots,J_{l-1}^k}{(\phi,\mathbf{w},\mathbf{A})}$ is obtained for $J_1^k=J_{h_1}^k, J_2^k=J_{h_2}^k,\dots,J_l^k=J_{h_l}^k$. i.e
$\max_{J_1^k,\dots,J_l^k}[D_k\overset{J_1^k,\dots,J_l^k}{(\phi,\mathbf{w},\mathbf{A})}] = \frac{W_{J_{h_1}^k}}{W_n}\phi((a+b)I-\frac{1}{W_{J_{h_1}^k}}\sum_{i\in J_{h_1}^k}w_iA_i)+\frac{W_{J_{h_2}^k}}{W_n}\phi((a+b)I-\frac{1}{W_{J_{h_2}^k}}\sum_{i\in J_{h_2}^k}w_iA_i)+\dots+\frac{W_{J_{h_l}^k}}{W_n}\phi((a+b)I-\frac{1}{W_{J_{h_l}^k}}\sum_{i\in J_{h_l}^k}w_iA_i)$
$=\frac{W_{J_{h_1}^k}}{W_n}\phi(\frac{1}{W_{J_{h_1}^k}}\sum_{i\in J_{h_1}^k}w_i((a+b)I-A_i))+\frac{W_{J_{h_2}^k}}{W_n}\phi(\frac{1}{W_{J_{h_2}^k}}\sum_{i\in J_{h_2}^k}w_i((a+b)I-A_i))+\dots+\frac{W_{J_{h_l}^k}}{W_n}\phi(\frac{1}{W_{J_{h_l}^k}}\sum_{i\in J_{h_l}^k}w_i((a+b)I-A_i))$
$\leq\frac{1}{w_n}\sum_{i\in J_{h_l}^k}w_i(\phi(aI)+\phi(bI)-\phi(A_i))+\frac{1}{w_n}\sum_{i\in J_{h_2}^k}w_i(\phi(aI)+\phi(bI)-\phi(A_i))+\dots+\frac{1}{w_n}\sum_{i\in J_{h_l}^k}w_i(\phi(aI)+\phi(bI)-\phi(A_i))$

$= \frac{1}{w_n} \sum_{i \in J^k_{h_1} \cup \ldots \cup J^k_{h_l}} w_i(\phi(aI) + \phi(bI) - \phi(A_i))$
$= \frac{1}{w_n} \sum_{i=1}^{n} w_i(\phi(aI) + \phi(bI) - \phi(A_i)).$
Hence it is proved that

$$\phi(aI) + \phi(bI) - \frac{1}{W_n} \sum_{i=1}^{n} w_i \phi(A_i) \geq M_k.$$

Now we have to prove that

$$M_2 \geq \phi((a+b)I - \frac{1}{W_n} \sum_{i=1}^{n} w_i A_i).$$

For this we assume that maximum of the funtional $D_2(\overset{J^2_1, J^2_2}{\phi}, \mathbf{w}, \mathbf{A})$ is obtained for $J^2_1 = J^k_{q_1}$ and $J^2_2 = J^2_{q_2}$ i.e

$\max_{J^2_1, J^2_2}[D_2(\overset{J^2_1, J^2_2}{\phi}, \mathbf{w}, \mathbf{A})]$
$= \frac{W_{J^2_{q_1}}}{W_n} \phi((a+b)I - \frac{1}{W_{J^2_{q_1}}} \sum_{i \in J^2_{q_1}} w_i A_i) + \frac{W_{J^2_{q_2}}}{W_n} \phi((a+b)I - \frac{1}{W_{J^2_{q_2}}} \sum_{i \in J^2_{q_2}} w_i A_i)$
$= \frac{W_{J^2_{q_1}}}{W_n} \phi(\frac{1}{W_{J^2_{q_1}}} \sum_{i \in J^2_{q_1}} w_i((a+b)I - A_i)) + \frac{W_{J^2_{q_2}}}{W_n} \phi(\frac{1}{W_{J^2_{q_2}}} \sum_{i \in J^2_{q_2}} w_i((a+b)I - A_i))$
$\geq \phi(\frac{1}{W_n} \sum_{i \in J^2_{q_1}} w_i((a+b)I - A_i) + \frac{1}{W_n} \sum_{i \in J^2_{q_2}} w_i((a+b)I - A_i))$
$= \phi(\frac{1}{W_n} \sum_{i \in J^2_{q_1} \cup J^2_{q_2}} w_i((a+b)I - A_i))$
$= \phi(\frac{1}{W_n} \sum_{i=1}^{n} w_i((a+b)I - A_i).$
Hence

$$M_2 \geq \phi((a+b)I - \frac{1}{W_n} \sum_{i=1}^{n} w_i A_i).$$

□

Theorem 2.3. *Let* ***A*** *be an n-tuple of selfadjoint operators defined on $[a,b] \subset \mathbb{R}$ with $w_i > 0$; $i = 1, 2, ..., n$ such that $W_n = \sum_{i=1}^{n} w_i$. If $\phi : [a,b] \to \mathbb{R}$ is an operator-convex function, then we have*

$$\begin{aligned} &\phi(aI) + \phi(bI) - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i \phi(A_i) \geq H_k \geq H_{k-1} \geq ... \geq H_3 \geq H_2 \geq \\ &\geq \phi((a+b)I - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i A_i) \end{aligned} \tag{2. 6}$$

where

$$H_k := \min_{J^k_1, \ldots, J^k_l} [D_k(\overset{J^k_1, \ldots, J^k_l}{\phi}, \boldsymbol{w}, \boldsymbol{A})]$$

If f is an operator-concave function then (2. 6) *is reversed.*

Proof. Its proof is similar to Theorem 2.2 but we use the minimum of the functional for l index sets instead functionals for $l-1$ index set. □

3. APPLICATIONS TO OPERATOR POWER MEANS

Mixed-symmetric means according to (2. 2), (2. 5) and (2. 6) are defined as

$$M(r,s;\mathbf{w},\mathbf{A}) := (W_J(\frac{1}{W_J}\sum_{i\in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}},$$

$$M_n(s,r;\mathbf{w},\mathbf{A};k) := (\max_{J_1^k,\ldots,J_l^k}\sum_{j=1}^{l}\frac{W_{J_j^k}}{W_n}((a+b)I - \frac{1}{W_{J_j^k}}\sum_{i\in J_j^k} w_i {A_i}^r)^{\frac{s}{r}})^{\frac{1}{s}}$$

and

$$H_n(s,r;\mathbf{w},\mathbf{A};k) := (\min_{J_1^k,\ldots,J_l^k}\sum_{j=1}^{l}\frac{W_{J_j^k}}{W_n}((a+b)I - \frac{1}{W_{J_j^k}}\sum_{i\in J_j^k} w_i {A_i}^r)^{\frac{s}{r}})^{\frac{1}{s}},$$

where $r,s\in\mathbb{R}\setminus\{0\}$ and $1\le l\le k\le n$.
The idea of above means for convex functions is given in [1].

Corollary 3.1. *Let* $\boldsymbol{A}$ *be* n*-tuple of strictly positive operators with* $w_i>0$*;* $i=1,2,...,n$ *such that* $W_n=\sum_{i=1}^{n} w_i = 1$*. Then for any nonempty proper subset* J *of* $\{1,2,...,n\}$*, we have*

$$M_s(\boldsymbol{w},\boldsymbol{A}) \le M(r,s;\boldsymbol{w},\boldsymbol{A}) \le M_r(\boldsymbol{w},\boldsymbol{A}),$$

if either

(i) $1\le s\le r$ *or*
(ii) $-r\le s\le -1$ *or*
(iii) $r\ge s\le 2r$*,* $s\le -1$

holds. While (3. 7) *is reversed if either*

(iv) $r\le s\le -1$ *or*
(v) $1\le s\le -r$ *or*
(vi) $r\le s\le 2r$*,* $s\ge 1$ *holds.*

Proof. There are basically six cases to prove this theorem
Case 1: Let us suppose $1\le s\le r$ then we have $0\le \frac{s}{r}\le 1$. Now by using Theorem 2.1 for a concave function ϕ, we have

$$\sum_{i=1}^{n} w_i\phi(A_i) \le W_J\phi(\frac{1}{W_J}\sum_{i\in J} w_iA_i) + W_{\bar{J}}\phi(\frac{1}{W_{\bar{J}}}\sum_{i\in\bar{J}} w_iA_i) \le \phi(\sum_{i=1}^{n} w_iA_i). \quad (3.\ 7)$$

Since $\phi(x) = x^p$ is concave function for $0 < p \leq 1$, by putting $A_i = A_i^r$ in (3. 7), we get

$$\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}} \leq W_J (\frac{1}{W_J} \sum_{i \in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}} (\frac{1}{W_{\bar{J}}} \sum_{i \in \bar{J}} w_i A_i^r)^{\frac{s}{r}} \leq (\sum_{i=1}^{n} w_i A_i^r)^{\frac{s}{r}}.$$

Since $f(x) = x^{\frac{1}{p}}$ is an increasing function for $p \geq 1$, from above inequality we have

$$(\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq (W_J (\frac{1}{W_J} \sum_{i \in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}} (\frac{1}{W_{\bar{J}}} \sum_{i \in \bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq ((\sum_{i=1}^{n} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}},$$

which implies that

$$(\sum_{i=1}^{n} w_i A_i^s)^{\frac{1}{s}} \leq (W_J (\frac{1}{W_J} \sum_{i \in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}} (\frac{1}{W_{\bar{J}}} \sum_{i \in \bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq (\sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}}.$$

Hence

$$M_s(\mathbf{w}, \mathbf{A}) \leq M(r, s; \mathbf{w}, \mathbf{A}) \leq M_r(\mathbf{w}, \mathbf{A}).$$

Case 2: Let us suppose $-r \leq s \leq -1$. Then we have $-1 \leq \frac{s}{r} < 0$. Now by using Theorem 2.1 for a convex function ϕ, we have

$$\sum_{i=1}^{n} w_i \phi(A_i) \geq W_J \phi(\frac{1}{W_J} \sum_{i \in J} w_i A_i) + W_{\bar{J}} \phi(\frac{1}{W_{\bar{J}}} \sum_{i \in \bar{J}} w_i A_i) \geq \phi(\sum_{i=1}^{n} w_i A_i). \quad (3.\ 8)$$

Since $\phi(x) = x^p$ is a convex function for $-1 \leq p < 0$, by putting $A_i = A_i^r$ in (3. 8), we get

$$\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}} \geq W_J (\frac{1}{W_J} \sum_{i \in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}} (\frac{1}{W_{\bar{J}}} \sum_{i \in \bar{J}} w_i A_i^r)^{\frac{s}{r}} \geq (\sum_{i=1}^{n} w_i A_i^r)^{\frac{s}{r}}.$$

Since $f(x) = x^{\frac{1}{p}}$ is a decreasing function for $p \leq -1$, from above inequality we have

$$(\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq (W_J (\frac{1}{W_J} \sum_{i \in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}} (\frac{1}{W_{\bar{J}}} \sum_{i \in \bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq ((\sum_{i=1}^{n} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}},$$

which implies that

$$(\sum_{i=1}^{n} w_i A_i^s)^{\frac{1}{s}} \leq (W_J (\frac{1}{W_J} \sum_{i \in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}} (\frac{1}{W_{\bar{J}}} \sum_{i \in \bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq (\sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}}.$$

Hence

$$M_s(\mathbf{w}, \mathbf{A}) \leq M(r, s; \mathbf{w}, \mathbf{A}) \leq M_r(\mathbf{w}, \mathbf{A}).$$

Case 3: Let us suppose $r \geq s \geq 2r, s \leq -1$ then we have $1 \leq \frac{s}{r} \leq 2$ where $s \leq -1$. Since $\phi(x) = x^p$ is convex function for $1 \leq p \leq 2$, by putting $A_i = A_i^r$ in (3. 8), we get

$$\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}} \geq W_J (\frac{1}{W_J} \sum_{i \in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}} (\frac{1}{W_{\bar{J}}} \sum_{i \in \bar{J}} w_i A_i^r)^{\frac{s}{r}} \geq (\sum_{i=1}^{n} w_i A_i^r)^{\frac{s}{r}}.$$

Since $f(x) = x^{\frac{1}{p}}$ is a decreasing function for $p \leq -1$, from above inequality we have

$$(\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq (W_J(\frac{1}{W_J}\sum_{i\in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq ((\sum_{i=1}^{n} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}},$$

which implies that

$$(\sum_{i=1}^{n} w_i A_i^s)^{\frac{1}{s}} \leq (W_J(\frac{1}{W_J}\sum_{i\in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq (\sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}}.$$

Hence

$$M_s(\mathbf{w}, \mathbf{A}) \leq M(r, s; \mathbf{w}, \mathbf{A}) \leq M_r(\mathbf{w}, \mathbf{A}).$$

Case 4: Let us suppose $r \leq s \leq -1$ then we have $0 < \frac{s}{r} \leq 1$. Now by using Theorem 2.1 for a concave function ϕ, we have

$$\sum_{i=1}^{n} w_i \phi(A_i) \leq W_J \phi(\frac{1}{W_J}\sum_{i\in J} w_i A_i) + W_{\bar{J}} \phi(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i) \leq \phi(\sum_{i=1}^{n} w_i A_i). \quad (3.\ 9)$$

Since $\phi(x) = x^p$ is concave function for $0 < p \leq 1$, by putting $A_i = A_i^r$ in (3. 9), we get

$$\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}} \leq W_J(\frac{1}{W_J}\sum_{i\in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^r)^{\frac{s}{r}} \leq (\sum_{i=1}^{n} w_i A_i^r)^{\frac{s}{r}}.$$

Since $f(x) = x^{\frac{1}{p}}$ is a decreasing function for $p \leq -1$, from above inequality we have

$$(\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \geq (W_J(\frac{1}{W_J}\sum_{i\in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \geq ((\sum_{i=1}^{n} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}},$$

which implies that

$$(\sum_{i=1}^{n} w_i A_i^s)^{\frac{1}{s}} \geq (W_J(\frac{1}{W_J}\sum_{i\in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \geq (\sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}}.$$

Hence

$$M_s(\mathbf{w}, \mathbf{A}) \geq M(r, s; \mathbf{w}, \mathbf{A}) \geq M_r(\mathbf{w}, \mathbf{A}).$$

Case 5: Let us suppose $1 \leq s \leq -r$ then we have $-1 \leq \frac{s}{r} < 0$. Now by using Theorem 2.1 for a convex function ϕ, we have

$$\sum_{i=1}^{n} w_i \phi(A_i) \geq W_J \phi(\frac{1}{W_J}\sum_{i\in J} w_i A_i) + W_{\bar{J}} \phi(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i) \geq \phi(\sum_{i=1}^{n} w_i A_i). \quad (3.\ 10)$$

Since $\phi(x) = x^p$ is convex function for $-1 \leq \frac{s}{r} < 0$, by putting $A_i = A_i^r$ in (3. 10), we get

$$\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}} \geq W_J(\frac{1}{W_J}\sum_{i\in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^r)^{\frac{s}{r}} \geq (\sum_{i=1}^{n} w_i A_i^r)^{\frac{s}{r}}.$$

Since $f(x) = x^{\frac{1}{p}}$ is an increasing function for $p \geq 1$, from above inequality we have

$$(\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \geq (W_J(\frac{1}{W_J}\sum_{i\in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in\bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \geq ((\sum_{i=1}^{n} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}},$$

which implies that

$$(\sum_{i=1}^{n} w_i A_i^s)^{\frac{1}{s}} \geq (W_J(\frac{1}{W_J}\sum_{i\in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in\bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \geq (\sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}}.$$

Hence

$$M_s(\mathbf{w}, \mathbf{A}) \geq M(r, s; \mathbf{w}, \mathbf{A}) \geq M_r(\mathbf{w}, \mathbf{A}).$$

Case 6: Let us suppose $r \leq s \leq 2r, s \geq 1$ then we have $1 \leq \frac{s}{r} \leq 2$ where $s \geq 1$. Since $\phi(x) = x^p$ is convex function for $1 \leq p \leq 2$, by putting $A_i = A_i^r$ in (3. 10), we get

$$\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}} \geq W_J(\frac{1}{W_J}\sum_{i\in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in\bar{J}} w_i A_i^r)^{\frac{s}{r}} \geq (\sum_{i=1}^{n} w_i A_i^r)^{\frac{s}{r}}.$$

Since $f(x) = x^{\frac{1}{p}}$ is an increasing function for $p \geq 1$, from above inequality we have

$$(\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \geq (W_J(\frac{1}{W_J}\sum_{i\in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in\bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \geq ((\sum_{i=1}^{n} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}},$$

which implies that

$$(\sum_{i=1}^{n} w_i A_i^s)^{\frac{1}{s}} \geq (W_J(\frac{1}{W_J}\sum_{i\in J} w_i A_i^r)^{\frac{s}{r}} + W_{\bar{J}}(\frac{1}{W_{\bar{J}}}\sum_{i\in\bar{J}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \geq (\sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}}.$$

Hence

$$M_s(\mathbf{w}, \mathbf{A}) \geq M(r, s; \mathbf{w}, \mathbf{A}) \geq M_r(\mathbf{w}, \mathbf{A}).$$

□

Corollary 3.2. *Let* $[a, b] \subset \mathbb{R}$ *and* ***A*** *be* n*-tuple of strictly positive operators with* $w_i > 0$*;* $i = 1, 2, ..., n$ *such that* $W_n = \sum_{i=1}^{n} w_i$*. Then we have*

$$((a+b)I - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i {A_i}^r)^{\frac{1}{r}} \geq M_n(s, r; \mathbf{w}, \mathbf{A}; 2) \geq ... \geq M_n(s, r; \mathbf{w}, \mathbf{A}; k) \geq$$
$$\geq ((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i {A_i}^s)^{\frac{1}{s}} \quad (3.\ 11)$$

if either

(i) $1 \leq s \leq r$ *or*
(ii) $-r \leq s \leq -1$ *or*
(iii) $r \geq s \leq 2r$, $s \leq -1$

holds. While (3. 11) *is reversed if either*

(iv) $r \leq s \leq -1$ *or*
(v) $1 \leq s \leq -r$ *or*

(vi) $r \leq s \leq 2r$, $s \geq 1$ *holds.*

Proof. There are basically six cases to prove this theorem
Case 1: Let us suppose $1 \leq s \leq r$ then we have $0 \leq \frac{s}{r} \leq 1$. Now by using Theorem 2.2 for a concave function ϕ, we have

$$\begin{aligned} &\phi(aI) + \phi(bI) - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i\phi(A_i) \leq M_k \leq M_{k-1} \leq ... \leq M_3 \leq M_2 \leq \\ &\leq \phi((a+b)I - \tfrac{1}{W_n}\sum_{i=1}^{n} w_iA_i). \end{aligned} \tag{3. 12}$$

Since

$$\phi(x) = x^p$$

is concave function for $0 < p \leq 1$, by putting $A_i = A_i^r$ in (3. 12), we get
$(aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n}\sum_{i=1}^{n} w_i(A_i^r)^{\frac{s}{r}} \leq \max_{J_1^k,...,J_l^k}\sum_{j=1}^{l}\frac{W_{J_j^k}}{W_n}((a+b)I - \frac{1}{W_{J_j^k}}\sum_{i\in J_j^k} w_iA_i^r)^{\frac{s}{r}} \leq \max_{J_1^{k-1},...,J_l^{k-1}}\sum_{j=1}^{l}\frac{W_{J_j^{k-1}}}{W_n}((a+b)I - \frac{1}{W_{J_j^{k-1}}}\sum_{i\in J_j^{k-1}} w_iA_i^r)^{\frac{s}{r}} \leq ... \leq \max_{J_1^3,...,J_l^3}\sum_{j=1}^{l}\frac{W_{J_j^3}}{W_n}((a+b)I - \frac{1}{W_{J_j^3}}\sum_{i\in J_j^3} w_iA_i^r)^{\frac{s}{r}} \leq \max_{J_1^2,J_l^2}\sum_{j=1}^{l}\frac{W_{J_j^2}}{W_n}((a+b)I - \frac{1}{W_{J_j^2}}\sum_{i\in J_j^2} w_iA_i^r)^{\frac{s}{r}} \leq ((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_iA_i^r)^{\frac{1}{r}}.$
Since

$$f(x) = x^{\frac{1}{p}}; \quad p \geq 1$$

is an increasing function, from above inequality we have

$$\begin{aligned} &((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i(A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq \\ &\leq (\max_{J_1^k,...,J_l^k}\sum_{j=1}^{l}\tfrac{W_{J_j^k}}{W_n}((a+b)I - \tfrac{1}{W_{J_j^k}}\sum_{i\in J_j^k} w_iA_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\leq (\max_{J_1^{k-1},...,J_l^{k-1}}\sum_{j=1}^{l}\tfrac{W_{J_j^{k-1}}}{W_n}((a+b)I - \tfrac{1}{W_{J_j^{k-1}}}\sum_{i\in J_j^{k-1}} w_iA_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq \\ &\vdots \\ &\leq (\max_{J_1^3,...,J_l^3}\sum_{j=1}^{l}\tfrac{W_{J_j^3}}{W_n}((a+b)I - \tfrac{1}{W_{J_j^3}}\sum_{i\in J_j^3} w_iA_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\leq (\max_{J_1^2,J_l^2}\sum_{j=1}^{l}\tfrac{W_{J_j^2}}{W_n}((a+b)I - \tfrac{1}{W_{J_j^2}}\sum_{i\in J_j^2} w_iA_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq (((a+b)I - \tfrac{1}{W_n}\sum_{i=1}^{n} w_iA_i^r)^{\frac{1}{r}})^{\frac{1}{s}}. \end{aligned}$$

Hence
$((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n}\sum_{i=1}^{n} w_i(A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq M_n(s,r;\mathbf{w},\mathbf{A};k) \leq M_n(s,r;\mathbf{w},\mathbf{A};k-1) \leq ... \leq M_n(s,r;\mathbf{w},\mathbf{A};3) \leq M_n(s,r;\mathbf{w},\mathbf{A};2) \leq ((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_i{A_i}^r)^{\frac{1}{r}}.$

Case 2: Suppose that $-r \leq s \leq -1$ then we have $-1 \leq \frac{s}{r} < 0$. Now by using Theorem 2.2 for a convex function ϕ, we have

$$\begin{aligned} &\phi(aI) + \phi(bI) - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i \phi(A_i) \geq M_k \geq M_{k-1} \geq ... \geq M_3 \geq M_2 \geq \\ &\geq \phi((a+b)I - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i A_i). \end{aligned} \tag{3. 13}$$

Since

$$\phi(x) = x^p$$

is convex function for $-1 \leq p < 0$, by putting $A_i = A_i^r$ in (3. 13), we get
$(aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n} \sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}} \geq \max_{J_1^k,\dots,J_l^k} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} ((a+b)I - \frac{1}{W_{J_j^k}} \sum_{i\in J_j^k} w_i A_i^r)^{\frac{s}{r}} \geq \max_{J_1^{k-1},\dots,J_l^{k-1}} \sum_{j=1}^{l} \frac{W_{J_j^{k-1}}}{W_n} ((a+b)I - \frac{1}{W_{J_j^{k-1}}} \sum_{i\in J_j^{k-1}} w_i A_i^r)^{\frac{s}{r}} \geq ... \geq \max_{J_1^3,\dots,J_l^3} \sum_{j=1}^{l} \frac{W_{J_j^3}}{W_n} ((a+b)I - \frac{1}{W_{J_j^3}} \sum_{i\in J_j^3} w_i A_i^r)^{\frac{s}{r}} \geq \max_{J_1^2, J_l^2} \sum_{j=1}^{l} \frac{W_{J_j^2}}{W_n} ((a+b)I - \frac{1}{W_{J_j^2}} \sum_{i\in J_j^2} w_i A_i^r)^{\frac{s}{r}} \geq ((a+b)I - \frac{1}{W_n} \sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}}$.
Since

$$f(x) = x^{\frac{1}{p}}; \quad p \leq -1$$

is an decreasing function, from above inequality we have

$$\begin{aligned} &((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\leq (\max_{J_1^k,\dots,J_l^k} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} ((a+b)I - \tfrac{1}{W_{J_j^k}} \sum_{i\in J_j^k} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\leq (\max_{J_1^{k-1},\dots,J_l^{k-1}} \sum_{j=1}^{l} \frac{W_{J_j^{k-1}}}{W_n} ((a+b)I - \tfrac{1}{W_{J_j^{k-1}}} \sum_{i\in J_j^{k-1}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\vdots \\ &\leq (\max_{J_1^3,\dots,J_l^3} \sum_{j=1}^{l} \frac{W_{J_j^3}}{W_n} ((a+b)I - \tfrac{1}{W_{J_j^3}} \sum_{i\in J_j^3} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\leq (\max_{J_1^2, J_l^2} \sum_{j=1}^{l} \frac{W_{J_j^2}}{W_n} ((a+b)I - \tfrac{1}{W_{J_j^2}} \sum_{i\in J_j^2} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\leq (((a+b)I - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}})^{\frac{1}{s}}. \end{aligned}$$

Hence
$((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n} \sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq M_n(s, r; \mathbf{w}, \mathbf{A}; k) \leq M_n(s, r; \mathbf{w}, \mathbf{A}; k-1) \leq ... \leq M_n(s, r; \mathbf{w}, \mathbf{A}; 3) \leq M_n(s, r; \mathbf{w}, \mathbf{A}; 2) \leq ((a+b)I - \frac{1}{W_n} \sum_{i=1}^{n} w_i {A_i}^r)^{\frac{1}{r}}$.

Case 3: Let us suppose $r \geq s \geq 2r, s \leq -1$ then we have $1 \leq \frac{s}{r} \leq 2$ where $s \leq -1$. Now by using Theorem 2.2 for a convex function ϕ, we have

$$\begin{aligned} &\phi(aI) + \phi(bI) - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i \phi(A_i) \geq M_k \geq M_{k-1} \geq ... \geq M_3 \geq M_2 \geq \\ &\geq \phi((a+b)I - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i A_i). \end{aligned} \tag{3. 14}$$

Since

$$\phi(x) = x^p$$

is convex function for $1 \leq p \leq 2$, by putting $A_i = A_i^r$ in (3. 14), we get $(aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n} \sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}} \geq \underset{J_1^k,...,J_l^k}{max} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} ((a+b)I - \frac{1}{W_{J_j^k}} \sum_{i \in J_j^k} w_i A_i^r)^{\frac{s}{r}} \geq \underset{J_1^{k-1},...,J_l^{k-1}}{max} \sum_{j=1}^{l} \frac{W_{J_j^{k-1}}}{W_n} ((a+b)I - \frac{1}{W_{J_j^{k-1}}} \sum_{i \in J_j^{k-1}} w_i A_i^r)^{\frac{s}{r}} \geq ... \geq \underset{J_1^3,...,J_l^3}{max} \sum_{j=1}^{l} \frac{W_{J_j^3}}{W_n} ((a+b)I - \frac{1}{W_{J_j^3}} \sum_{i \in J_j^3} w_i A_i^r)^{\frac{s}{r}} \geq \underset{J_1^2,J_l^2}{max} \sum_{j=1}^{l} \frac{W_{J_j^2}}{W_n} ((a+b)I - \frac{1}{W_{J_j^2}} \sum_{i \in J_j^2} w_i A_i^r)^{\frac{s}{r}} \geq ((a+b)I - \frac{1}{W_n} \sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}}$.

Since

$$f(x) = x^{\frac{1}{p}}; \quad p \leq -1$$

is an decreasing function, from above inequality we have

$$\begin{aligned} &((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\leq (\underset{J_1^k,...,J_l^k}{max} \sum_{j=1}^{l} \tfrac{W_{J_j^k}}{W_n} ((a+b)I - \tfrac{1}{W_{J_j^k}} \sum_{i \in J_j^k} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\leq (\underset{J_1^{k-1},...,J_l^{k-1}}{max} \sum_{j=1}^{l} \tfrac{W_{J_j^{k-1}}}{W_n} ((a+b)I - \tfrac{1}{W_{J_j^{k-1}}} \sum_{i \in J_j^{k-1}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\leq \\ &\vdots \\ &\leq (\underset{J_1^3,...,J_l^3}{max} \sum_{j=1}^{l} \tfrac{W_{J_j^3}}{W_n} ((a+b)I - \tfrac{1}{W_{J_j^3}} \sum_{i \in J_j^3} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\leq (\underset{J_1^2,J_l^2}{max} \sum_{j=1}^{l} \tfrac{W_{J_j^2}}{W_n} ((a+b)I - \tfrac{1}{W_{J_j^2}} \sum_{i \in J_j^2} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\leq (((a+b)I - \tfrac{1}{W_n} \sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}})^{\frac{1}{s}}. \end{aligned}$$

Hence

$((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n} \sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \leq M_n(s, r; \mathbf{w}, \mathbf{A}; k) \leq M_n(s, r; \mathbf{w}, \mathbf{A}; k-1) \leq ... \leq M_n(s, r; \mathbf{w}, \mathbf{A}; 3) \leq M_n(s, r; \mathbf{w}, \mathbf{A}; 2) \leq ((a+b)I - \frac{1}{W_n} \sum_{i=1}^{n} w_i {A_i}^r)^{\frac{1}{r}}$.

Case 4: Suppose that $r \leq s \leq -1$ then we have $0 < \frac{s}{r} \leq 1$. Now by using Theorem 2.2 for a concave function ϕ, we have

$$\begin{aligned} &\phi(aI) + \phi(bI) - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i\phi(A_i) \leq M_k \leq M_{k-1} \leq ... \leq M_3 \leq M_2 \leq \\ &\leq \phi((a+b)I - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i A_i). \end{aligned} \tag{3. 15}$$

Since

$$\phi(x) = x^p$$

is concave function for $0 < p \leq 1$, by putting $A_i = A_i^r$ in (3. 15), we get
$(aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n}\sum_{i=1}^{n} w_i(A_i^r)^{\frac{s}{r}} \leq \underset{J_1^k,\ldots,J_l^k}{max}\sum_{j=1}^{l}\frac{W_{J_j^k}}{W_n}((a+b)I - \frac{1}{W_{J_j^k}}\sum_{i\in J_j^k} w_iA_i^r)^{\frac{s}{r}} \leq \underset{J_1^{k-1},\ldots,J_l^{k-1}}{max}\sum_{j=1}^{l}\frac{W_{J_j^{k-1}}}{W_n}((a+b)I - \frac{1}{W_{J_j^{k-1}}}\sum_{i\in J_j^{k-1}} w_iA_i^r)^{\frac{s}{r}} \leq ... \leq \underset{J_1^3,\ldots,J_l^3}{max}\sum_{j=1}^{l}\frac{W_{J_j^3}}{W_n}((a+b)I - \frac{1}{W_{J_j^3}}\sum_{i\in J_j^3} w_iA_i^r)^{\frac{s}{r}} \leq \underset{J_1^2,J_l^2}{max}\sum_{j=1}^{l}\frac{W_{J_j^2}}{W_n}((a+b)I - \frac{1}{W_{J_j^2}}\sum_{i\in J_j^2} w_iA_i^r)^{\frac{s}{r}} \leq ((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_iA_i^r)^{\frac{1}{r}}$.
Since

$$f(x) = x^{\frac{1}{p}}; \quad p \leq -1$$

is an decreasing function, from above inequality we have

$$\begin{aligned} &((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i(A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\geq (\underset{J_1^k,\ldots,J_l^k}{max}\sum_{j=1}^{l}\frac{W_{J_j^k}}{W_n}((a+b)I - \tfrac{1}{W_{J_j^k}}\sum_{i\in J_j^k} w_iA_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\geq (\underset{J_1^{k-1},\ldots,J_l^{k-1}}{max}\sum_{j=1}^{l}\frac{W_{J_j^{k-1}}}{W_n}((a+b)I - \tfrac{1}{W_{J_j^{k-1}}}\sum_{i\in J_j^{k-1}} w_iA_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\geq \\ &\vdots \\ &\geq (\underset{J_1^3,\ldots,J_l^3}{max}\sum_{j=1}^{l}\frac{W_{J_j^3}}{W_n}((a+b)I - \tfrac{1}{W_{J_j^3}}\sum_{i\in J_j^3} w_iA_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\geq (\underset{J_1^2,J_l^2}{max}\sum_{j=1}^{l}\frac{W_{J_j^2}}{W_n}((a+b)I - \tfrac{1}{W_{J_j^2}}\sum_{i\in J_j^2} w_iA_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\ &\geq (((a+b)I - \tfrac{1}{W_n}\sum_{i=1}^{n} w_iA_i^r)^{\frac{1}{r}})^{\frac{1}{s}}. \end{aligned}$$

Hence
$((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n}\sum_{i=1}^{n} w_i(A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \geq M_n(s,r;\mathbf{w},\mathbf{A};k) \geq M_n(s,r;\mathbf{w},\mathbf{A};k-1) \geq ... \geq M_n(s,r;\mathbf{w},\mathbf{A};3) \geq M_n(s,r;\mathbf{w},\mathbf{A};2) \geq ((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_i{A_i}^r)^{\frac{1}{r}}$.

Case 5: Let us suppose $1 \leq s \leq -r$ then we have $-1 \leq \frac{s}{r} < 0$. Now by using Theorem 2.2 for a convex function ϕ, we have

$$\begin{aligned} &\phi(aI) + \phi(bI) - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i\phi(A_i) \geq M_k \geq M_{k-1} \geq ... \geq M_3 \geq M_2 \geq \\ &\geq \phi((a+b)I - \tfrac{1}{W_n}\sum_{i=1}^{n} w_iA_i). \end{aligned} \tag{3. 16}$$

Since

$$\phi(x) = x^p$$

is convex function for $-1 \leq \frac{s}{r} < 0$, by putting $A_i = A_i^r$ in (3. 16), we get

$$\begin{aligned} &(aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i(A_i^r)^{\frac{s}{r}} \\ &\geq \max_{J_1^m,\ldots,J_l^m} \sum_{j=1}^{l} \tfrac{W_{J_j^m}}{W_n}((a+b)I - \tfrac{1}{W_{J_j^m}}\sum_{i\in J_j^m} w_iA_i^r)^{\frac{s}{r}} \\ &\geq \max_{J_1^{m-1},\ldots,J_l^{m-1}} \sum_{j=1}^{l} \tfrac{W_{J_j^{m-1}}}{W_n}((a+b)I - \tfrac{1}{W_{J_j^{m-1}}}\sum_{i\in J_j^{m-1}} w_iA_i^r)^{\frac{s}{r}} \\ &\geq \\ &\vdots \\ &\geq \max_{J_1^3,\ldots,J_l^3} \sum_{j=1}^{l} \tfrac{W_{J_j^3}}{W_n}((a+b)I - \tfrac{1}{W_{J_j^3}}\sum_{i\in J_j^3} w_iA_i^r)^{\frac{s}{r}} \\ &\geq \max_{J_1^2,J_l^2} \sum_{j=1}^{l} \tfrac{W_{J_j^2}}{W_n}((a+b)I - \tfrac{1}{W_{J_j^2}}\sum_{i\in J_j^2} w_iA_i^r)^{\frac{s}{r}} \\ &\geq ((a+b)I - \tfrac{1}{W_n}\sum_{i=1}^{n} w_iA_i^r)^{\frac{1}{r}}. \end{aligned}$$

Since

$$f(x) = x^{\frac{1}{p}}; \quad p \geq 1$$

is an increasing function, from above inequality we have

$$
\begin{aligned}
&((aI)^{\frac{s}{r}}+(bI)^{\frac{s}{r}}-\tfrac{1}{W_n}\sum_{i=1}^{n} w_i(A_i^r)^{\frac{s}{r}})^{\frac{1}{s}}\\
&\geq (\max_{J_1^m,\dots,J_l^m}\sum_{j=1}^{l}\frac{W_{J_j^m}}{W_n}((a+b)I-\tfrac{1}{W_{J_j^m}}\sum_{i\in J_j^m} w_iA_i^r)^{\frac{s}{r}})^{\frac{1}{s}}\\
&\geq (\max_{J_1^{m-1},\dots,J_l^{m-1}}\sum_{j=1}^{l}\frac{W_{J_j^{m-1}}}{W_n}((a+b)I-\tfrac{1}{W_{J_j^{m-1}}}\sum_{i\in J_j^{m-1}} w_iA_i^r)^{\frac{s}{r}})^{\frac{1}{s}}\\
&\geq\\
&\vdots\\
&\geq (\max_{J_1^3,\dots,J_l^3}\sum_{j=1}^{l}\frac{W_{J_j^3}}{W_n}((a+b)I-\tfrac{1}{W_{J_j^3}}\sum_{i\in J_j^3} w_iA_i^r)^{\frac{s}{r}})^{\frac{1}{s}}\\
&\geq (\max_{J_1^2,J_l^2}\sum_{j=1}^{l}\frac{W_{J_j^2}}{W_n}((a+b)I-\tfrac{1}{W_{J_j^2}}\sum_{i\in J_j^2} w_iA_i^r)^{\frac{s}{r}})^{\frac{1}{s}}\\
&\geq (((a+b)I-\tfrac{1}{W_n}\sum_{i=1}^{n} w_iA_i^r)^{\frac{1}{r}})^{\frac{1}{s}}.
\end{aligned}
$$

Hence
$((aI)^{\frac{s}{r}}+(bI)^{\frac{s}{r}}-\frac{1}{W_n}\sum_{i=1}^{n} w_i(A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \geq M_n(s,r;\mathbf{w},\mathbf{A};k) \geq M_n(s,r;\mathbf{w},\mathbf{A};k-1) \geq$ $... \geq M_n(s,r;\mathbf{w},\mathbf{A};3) \geq M_n(s,r;\mathbf{w},\mathbf{A};2) \geq ((a+b)I-\frac{1}{W_n}\sum_{i=1}^{n} w_i{A_i}^r)^{\frac{1}{r}}$.

Case 6: Suppose that $r \leq s \leq 2r, s \geq 1$ then we have $1 \leq \frac{s}{r} \leq 2$ where $s \geq 1$. Now by using Theorem 2.2 for a convex function ϕ, we have

$$
\begin{aligned}
&\phi(aI)+\phi(bI)-\tfrac{1}{W_n}\sum_{i=1}^{n} w_i\phi(A_i) \geq M_k \geq M_{k-1} \geq ... \geq M_3 \geq M_2 \geq\\
&\geq \phi((a+b)I-\tfrac{1}{W_n}\sum_{i=1}^{n} w_iA_i).
\end{aligned}
\tag{3. 17}
$$

Since

$$\phi(x)=x^p$$

is convex function for $1 \leq p \leq 2$, by putting $A_i = A_i^r$ in (3. 17), we get

$$
\begin{aligned}
& (aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n}\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}} \\
& \geq \max_{J_1^m,\ldots,J_l^m} \sum_{j=1}^{l} \frac{W_{J_j^m}}{W_n}((a+b)I - \frac{1}{W_{J_j^m}} \sum_{i\in J_j^m} w_i A_i^r)^{\frac{s}{r}} \\
& \geq \max_{J_1^{m-1},\ldots,J_l^{m-1}} \sum_{j=1}^{l} \frac{W_{J_j^{m-1}}}{W_n}((a+b)I - \frac{1}{W_{J_j^{m-1}}} \sum_{i\in J_j^{m-1}} w_i A_i^r)^{\frac{s}{r}} \\
& \geq \\
& \ldots \\
& \geq \max_{J_1^3,\ldots,J_l^3} \sum_{j=1}^{l} \frac{W_{J_j^3}}{W_n}((a+b)I - \frac{1}{W_{J_j^3}} \sum_{i\in J_j^3} w_i A_i^r)^{\frac{s}{r}} \\
& \geq \max_{J_1^2,J_l^2} \sum_{j=1}^{l} \frac{W_{J_j^2}}{W_n}((a+b)I - \frac{1}{W_{J_j^2}} \sum_{i\in J_j^2} w_i A_i^r)^{\frac{s}{r}} \\
& \geq ((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}}.
\end{aligned}
$$

Since

$$f(x) = x^{\frac{1}{p}}; \quad p \geq 1$$

is an increasing function, from above inequality we have

$$
\begin{aligned}
& ((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n}\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\
& \geq (\max_{J_1^m,\ldots,J_l^m} \sum_{j=1}^{l} \frac{W_{J_j^m}}{W_n}((a+b)I - \frac{1}{W_{J_j^m}} \sum_{i\in J_j^m} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\
& \geq (\max_{J_1^{m-1},\ldots,J_l^{m-1}} \sum_{j=1}^{l} \frac{W_{J_j^{m-1}}}{W_n}((a+b)I - \frac{1}{W_{J_j^{m-1}}} \sum_{i\in J_j^{m-1}} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\
& \geq \\
& \vdots \\
& \geq (\max_{J_1^3,\ldots,J_l^3} \sum_{j=1}^{l} \frac{W_{J_j^3}}{W_n}((a+b)I - \frac{1}{W_{J_j^3}} \sum_{i\in J_j^3} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\
& \geq (\max_{J_1^2,J_l^2} \sum_{j=1}^{l} \frac{W_{J_j^2}}{W_n}((a+b)I - \frac{1}{W_{J_j^2}} \sum_{i\in J_j^2} w_i A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \\
& \geq (((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_i A_i^r)^{\frac{1}{r}})^{\frac{1}{s}}.
\end{aligned}
$$

Hence

$((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \frac{1}{W_n}\sum_{i=1}^{n} w_i (A_i^r)^{\frac{s}{r}})^{\frac{1}{s}} \geq M_n(s,r;\mathbf{w},\mathbf{A};k) \geq M_n(s,r;\mathbf{w},\mathbf{A};k-1) \geq$
$\ldots \geq M_n(s,r;\mathbf{w},\mathbf{A};3) \geq M_n(s,r;\mathbf{w},\mathbf{A};2) \geq ((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_i {A_i}^r)^{\frac{1}{r}}.$ □

Corollary 3.3. *Let* $[a,b] \subset \mathbb{R}$ *and* ***A*** *be an* n*-tuple of strictly positive operators with* $w_i > 0$*;* $i = 1,2,...,n$ *such that* $W_n = \sum_{i=1}^{n} w_i$*. Then we have*

$$\begin{aligned} &((a+b)I - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i {A_i}^r)^{\frac{1}{r}} \geq H_n(s,r;\mathbf{w},\mathbf{A};2) \geq ... \geq H_n(s,r;\mathbf{w},\mathbf{A};k) \\ &\geq ((aI)^{\frac{s}{r}} + (bI)^{\frac{s}{r}} - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i {A_i}^s)^{\frac{1}{s}}, \end{aligned} \tag{3. 18}$$

if either

(i) $1 \leq s \leq r$ *or*
(ii) $-r \leq s \leq -1$ *or*
(iii) $r \geq s \leq 2r$, $s \leq -1$

holds. While (3. 18) *is reversed if either*

(iv) $r \leq s \leq -1$ *or*
(v) $1 \leq s \leq -r$ *or*
(vi) $r \leq s \leq 2r$, $s \geq 1$ *holds.*

Proof. Its proof is similar to Theorem 3.2, but we use the minimum of the functional for l index sets instead functionals for $l-1$ index set. □

Now we introduce the following expressions for operator-convex functions in the same way as these are defined in [1] for convex functions:

$$g(\phi;p;\mathbf{A}) := \left\{ W_J \phi[\frac{1}{W_J}\sum_{i\in J} w_i A_i^p]^{\frac{1}{p}} + W_{\bar{J}} \phi[\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^p]^{\frac{1}{p}} \right\}^{\frac{1}{p}},$$

$$g_{k,n}(\phi;p;\mathbf{A}) := \left\{ \max_{J_1^k,...,J_l^k} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} \phi[((a+b)I - \frac{1}{W_{J_j^k}} \sum_{i\in J_l^k} w_i A_i^p)^{\frac{1}{p}}]^p \right\}^{\frac{1}{p}}$$

and

$$h_{k,n}(\phi;p;\mathbf{A}) := \left\{ \min_{J_1^k,...,J_l^k} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} \phi[((a+b)I - \frac{1}{W_{J_j^k}} \sum_{i\in J_l^k} w_i A_i^p)^{\frac{1}{p}}]^p \right\}^{\frac{1}{p}}.$$

Corollary 3.4. *Let* ***A*** *be an* n*-tuple of strictly positive operators with* $w_i > 0$*,* $i = 1,2,...,n$ *such that* $W_n = \sum_{i=1}^{n} w_i = 1$*,* ϕ *is a positive operator monotone function on* $S(0,\infty)$ *and* J *is any nonempty proper subset of* $\{1,2,...,n\}$*. Now if* $p \geq 1$*, then we have*

$$(\sum_{i=1}^{n} w_i \phi(A_i^p))^{\frac{1}{p}} \leq g(\phi;p;\mathbf{A}) \leq \phi((\sum_{i=1}^{n} w_i A_i^p)^{\frac{1}{p}}). \tag{3. 19}$$

If $p \leq -1$*, then* (3. 19) *is reversed.*

Proof. Let us suppose $p \geq 1$, now by using Theorem 2.1 for a concave function f, we have

$$\sum_{i=1}^{n} w_i f(A_i) \leq W_J f(\frac{1}{W_J}\sum_{i\in J} w_i A_i) + W_{\bar{J}} f(\frac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i) \leq f(\sum_{i=1}^{n} w_i A_i). \quad (3.\ 20)$$

Since the function $f(x) = \phi(x^{\frac{1}{p}})^p$ is operator-concave, where ϕ is a positive operator-monotone function defined on $S(0,\infty)$. Now by putting $A_i = A_i^p$ in (3. 20), we get

$$\begin{aligned}&\sum_{i=1}^{n} w_i \phi((A_i^p)^{\frac{1}{p}})^p \leq W_J \phi((\tfrac{1}{W_J}\sum_{i\in J} w_i A_i^p)^{\frac{1}{p}})^p \\ &+W_{\bar{J}} \phi((\tfrac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^p)^{\frac{1}{p}})^p \leq \phi((\sum_{i=1}^{n} w_i A_i^p)^{\frac{1}{p}})^p.\end{aligned} \quad (3.\ 21)$$

Since $g(x) = x^{\frac{1}{p}}$ is an increasing function for $p \geq 1$, we get

$$\begin{aligned}&(\sum_{i=1}^{n} w_i \phi((A_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}} \leq (W_J \phi((\tfrac{1}{W_J}\sum_{i\in J} w_i A_i^p)^{\frac{1}{p}})^p \\ &+W_{\bar{J}} \phi((\tfrac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}} \leq (\phi((\sum_{i=1}^{n} w_i A_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}},\end{aligned}$$

which implies that

$$\begin{aligned}&(\sum_{i=1}^{n} w_i \phi(A_i^p))^{\frac{1}{p}} \leq (W_J \phi((\tfrac{1}{W_J}\sum_{i\in J} w_i A_i^p)^{\frac{1}{p}})^p \\ &+W_{\bar{J}} \phi((\tfrac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}} \leq \phi((\sum_{i=1}^{n} w_i A_i^p)^{\frac{1}{p}}).\end{aligned}$$

Hence

$$(\sum_{i=1}^{n} w_i \phi(A_i^p))^{\frac{1}{p}} \leq g(\phi; p; \mathbf{A}) \leq \phi((\sum_{i=1}^{n} w_i A_i^p)^{\frac{1}{p}}).$$

Now we suppose that $p \leq -1$. Since $g(x) = x^{\frac{1}{p}}$ is decreasing function for $p \leq -1$, (3. 21) becomes

$$\begin{aligned}&(\sum_{i=1}^{n} w_i \phi((A_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}} \geq (W_J \phi((\tfrac{1}{W_J}\sum_{i\in J} w_i A_i^p)^{\frac{1}{p}})^p \\ &+W_{\bar{J}} \phi((\tfrac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}} \geq (\phi((\sum_{i=1}^{n} w_i A_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}},\end{aligned}$$

which implies that

$$\begin{aligned}&(\sum_{i=1}^{n} w_i \phi(A_i^p))^{\frac{1}{p}} \geq (W_J \phi((\tfrac{1}{W_J}\sum_{i\in J} w_i A_i^p)^{\frac{1}{p}})^p \\ &+W_{\bar{J}} \phi((\tfrac{1}{W_{\bar{J}}}\sum_{i\in \bar{J}} w_i A_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}} \geq \phi((\sum_{i=1}^{n} w_i A_i^p)^{\frac{1}{p}}).\end{aligned}$$

Hence

$$(\sum_{i=1}^{n} w_i \phi(A_i^p))^{\frac{1}{p}} \geq g(\phi; p; \mathbf{A}) \geq \phi((\sum_{i=1}^{n} w_i A_i^p)^{\frac{1}{p}}).$$

□

Corollary 3.5. *Let $I \subset \mathbb{R}$ be an interval and **A** be n-tuple of strictly positive operators with $w_i > 0$; $i = 1, 2, ..., n$ such that $W_n = \sum_{i=1}^{n} w_i$ and ϕ is a positive operator monotone function on $S(0, \infty)$. Now if $p \geq 1$, then we have*

$$(\phi((aI)^{\frac{1}{p}})^p + \phi((bI)^{\frac{1}{p}})^p - \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(A_i)^p)^{\frac{1}{p}} \leq g_{k,n}(\phi; p; \boldsymbol{A}) \leq ... \leq g_{2,n}(\phi; p; \boldsymbol{A}) \leq \phi(((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_i A_i{}^p)^{\frac{1}{p}}), \quad (3.\ 22)$$

where $1 \leq l \leq k \leq n$. If $p \leq -1$, then (3. 22) *is reversed.*

Proof. Let us suppose that $p \geq 1$, now by using Theorem 2.2 for a concave function ϕ, we have

$$\begin{aligned}
& f(aI) + f(bI) - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i f(A_i) \\
& \leq \max_{J_1^k,\ldots,J_l^k} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n} f((a+b)I - \tfrac{1}{W_{J_j^k}} \sum_{i\in J_l^k} w_i A_i) \\
& \leq \max_{J_1^{k-1},\ldots,J_l^{k-1}} \sum_{j=1}^{l} \frac{W_{J_j^{k-1}}}{W_n} f((a+b)I - \tfrac{1}{W_{J_j^{k-1}}} \sum_{i\in J_l^{k-1}} w_i A_i) \\
& \leq \\
& \vdots \\
& \leq \max_{J_1^3,\ldots,J_l^3} \sum_{j=1}^{l} \frac{W_{J_j^3}}{W_n} f((a+b)I - \tfrac{1}{W_{J_j^3}} \sum_{i\in J_l^3} w_i A_i) \\
& \leq \max_{J_1^2,J_l^2} \sum_{j=1}^{l} \frac{W_{J_j^2}}{W_n} f((a+b)I - \tfrac{1}{W_{J_j^2}} \sum_{i\in J_l^2} w_i A_i) \\
& \leq f((a+b)I - \tfrac{1}{W_n}\sum_{i=1}^{n} w_i A_i).
\end{aligned}$$

Since the function $f(x) = \phi(x^{\frac{1}{p}})^p$ is operator-concave, where ϕ is a positive operator-monotone function defined on $S(0, \infty)$. Now by putting $A_i = A_i^p$ in above inequality, we get

$$
\begin{aligned}
&\phi((aI)^{\frac{1}{p}})^p + \phi((bI)^{\frac{1}{p}})^p - \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi((A_i^p)^{\frac{1}{p}})^p) \\
&\leq \max_{J_1^k,\ldots,J_l^k} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n}\phi(((a+b)I - \frac{1}{W_{J_j^k}}\sum_{i\in J_l^k} w_iA_i^p)^{\frac{1}{p}})^p \\
&\leq \max_{J_1^{k-1},\ldots,J_l^{k-1}} \sum_{j=1}^{l} \frac{W_{J_j^{k-1}}}{W_n}\phi(((a+b)I - \frac{1}{W_{J_j^{k-1}}}\sum_{i\in J_l^{k-1}} w_iA_i^p)^{\frac{1}{p}})^p \\
&\leq \\
&\vdots \qquad\qquad\qquad\qquad (3.\ 23)\\
&\leq \max_{J_1^3,\ldots,J_l^3} \sum_{j=1}^{l} \frac{W_{J_j^3}}{W_n}\phi(((a+b)I - \frac{1}{W_{J_j^3}}\sum_{i\in J_l^3} w_iA_i^p)^{\frac{1}{p}})^p \\
&\leq \max_{J_1^2,J_l^2} \sum_{j=1}^{l} \frac{W_{J_j^2}}{W_n}\phi(((a+b)I - \frac{1}{W_{J_j^2}}\sum_{i\in J_l^2} w_iA_i^p)^{\frac{1}{p}})^p \\
&\leq \phi(((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_iA_i^p)^{\frac{1}{p}})^p.
\end{aligned}
$$

Since $g(x) = x^{\frac{1}{p}}$ is an increasing function for $p \geq 1$, (3. 23) becomes

$$
\begin{aligned}
&(\phi((aI)^{\frac{1}{p}})^p + \phi((bI)^{\frac{1}{p}})^p - \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi((A_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}} \\
&\leq (\max_{J_1^k,\ldots,J_l^k} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n}\phi(((a+b)I - \frac{1}{W_{J_j^k}}\textstyle\sum_{i\in J_l^k} w_iA_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}} \\
&\leq (\max_{J_1^{k-1},\ldots,J_l^{k-1}} \sum_{j=1}^{l} \frac{W_{J_j^{k-1}}}{W_n}\phi(((a+b)I - \frac{1}{W_{J_j^{k-1}}}\textstyle\sum_{i\in J_l^{k-1}} w_iA_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}} \\
&\leq \\
&\vdots \\
&\leq (\max_{J_1^3,\ldots,J_l^3} \sum_{j=1}^{l} \frac{W_{J_j^3}}{W_n}\phi(((a+b)I - \frac{1}{W_{J_j^3}}\textstyle\sum_{i\in J_l^3} w_iA_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}} \\
&\leq (\max_{J_1^2,J_l^2} \sum_{j=1}^{l} \frac{W_{J_j^2}}{W_n}\phi(((a+b)I - \frac{1}{W_{J_j^2}}\textstyle\sum_{i\in J_l^2} w_iA_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}} \\
&\leq (\phi(((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_iA_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}}.
\end{aligned}
$$

Hence

$(\phi((aI)^{\frac{1}{p}})^p + \phi((bI)^{\frac{1}{p}})^p - \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(A_i)^p)^{\frac{1}{p}} \leq g_{k,n}(\phi;p;\mathbf{A}) \leq g_{k-1,n}(\phi;p;\mathbf{A}) \leq \ldots \leq g_{3,n}(\phi;p;\mathbf{A}) \leq g_{2,n}(\phi;p;\mathbf{A}) \leq \phi(((a+b)I_{\frac{1}{W_n}}\sum_{i=1}^{n} w_iA_i{}^p)^{\frac{1}{p}})$.

Now we suppose that $p \leq -1$. Since $g(x) = x^{\frac{1}{p}}$ is decreasing function for $p \leq -1$,

(3. 23) becomes

$$(\phi((aI)^{\frac{1}{p}})^p + \phi((bI)^{\frac{1}{p}})^p - \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi((A_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}}$$
$$\geq (\max_{J_1^k,\dots,J_l^k} \sum_{j=1}^{l} \frac{W_{J_j^k}}{W_n}\phi(((a+b)I - \frac{1}{W_{J_j^k}}\sum_{i\in J_l^k} w_iA_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}}$$
$$\geq (\max_{J_1^{k-1},\dots,J_l^{k-1}} \sum_{j=1}^{l} \frac{W_{J_j^{k-1}}}{W_n}\phi(((a+b)I - \frac{1}{W_{J_j^{k-1}}}\sum_{i\in J_l^{k-1}} w_iA_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}}$$
$$\geq$$
$$\dots$$
$$\geq (\max_{J_1^3,\dots,J_l^3} \sum_{j=1}^{l} \frac{W_{J_j^3}}{W_n}\phi(((a+b)I - \frac{1}{W_{J_j^3}}\sum_{i\in J_l^3} w_iA_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}}$$
$$\geq (\max_{J_1^2,J_l^2} \sum_{j=1}^{l} \frac{W_{J_j^2}}{W_n}\phi(((a+b)I - \frac{1}{W_{J_j^2}}\sum_{i\in J_l^2} w_iA_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}}$$
$$\geq (\phi(((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_iA_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}}.$$

Hence

$$(\phi((aI)^{\frac{1}{p}})^p+\phi((bI)^{\frac{1}{p}})^p-\frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(A_i)^p)^{\frac{1}{p}} \geq g_{k,n}(\phi;p;\mathbf{A}) \geq g_{k-1,n}(\phi;p;\mathbf{A}) \geq \dots \geq$$
$$g_{3,n}(\phi;p;\mathbf{A}) \geq g_{2,n}(\phi;p;\mathbf{A}) \geq \phi(((a+b)I_{\frac{1}{W_n}}\sum_{i=1}^{n} w_i{A_i}^p)^{\frac{1}{p}}).$$ □

Corollary 3.6. *Let the conditions of Corollary* 3.5 *be satisfied. Then we have the following inequality*

$$(\phi((aI)^{\frac{1}{p}})^p+\phi((bI)^{\frac{1}{p}})^p-\frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(A_i)^p)^{\frac{1}{p}} \leq h_{k,n}(\phi;p;\boldsymbol{A}) \leq \dots \leq h_{2,n}(\phi;p;\boldsymbol{A}) \leq$$
$$\phi(((a+b)I - \frac{1}{W_n}\sum_{i=1}^{n} w_i{A_i}^p)^{\frac{1}{p}}), \quad (3.\ 24)$$

where $1 \leq l \leq k \leq n$*. If* $p \leq -1$*, then* (3. 24) *is reversed.*

Proof. Its proof is similar to the proof of Corollary 3.5. □

4. Acknowledgments

The authors are thankful to the referees for their valuable comments, guidance and patience.

References

[1] B. Mond and J. Pecaric, *Remarks on Jensen's inequality for operator convex functions*, Ann. Univ. Mariae Curie-Sklodowska Sec. A. **47**, No. 10 (1993) 96-103.
[2] T. Furuta, J. M. Hot, J. Pecaric and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Element, Zagreb 2005.
[3] S. S. Dragomir, *A refinement of Jensens inequality with applications for f-divergence measures*, Res. Rep. Coll. 10, (2007) (Preprint, Supp., Article 15).

[4] J. Baric, A. Matkovic and J. Pecaric, *variant of the Jensen-Mercer operator inequality for superquadratic functions*, Mathematical and Computer Modelling **51**, (2010) 1230-1239.

[5] S. S. Dragomir, *A new refinement of Jensens inequality in linear spaces with applications*, Mathematical and Computer Modelling **52**, (2010) 1497-1505.

[6] S. Ivelic, A. Matkovic and J. Pearic, *On a Jensen-Mercer operator inequality*, Banach J. Math. Anal. **5**, (2011) 19-28.

[7] M. Kian and M. S. Moslehian, *Refinements of the operator Jensen-Mercer inequality*, Electron. J. Linear Algebra **26**, (2013) 742-753.

[8] S. Iqbal, K. Krulic Himmelreich and J. Pecaric, *Refinements of Hardy-type Integral Inequalities with Kernels*, Punjab Univ. J. Math. Vol. **48**, No. 1 (2016) 19-28.

[9] Muhammad Muddassar and Ahsan Ali, *New Integral Inequalities Through Generalized Convex Functions*, Punjab Univ. J. Math. Vol. **46**, No. 2 (2014) 47-51.

[10] Muhammad Muddassar and Muhammad I. Bhatti, *Some Generalizations of Hermite-Hadamard Type Integral Inequalities and their Applications*, Punjab Univ. J. Math. Vol. **46**, No. 1 (2014) 9-18.

[11] Muhammad Adil Khan, Yousaf Khurshid, Tahir Ali and Nasir Rehman, *Inequalities for Three Times Differentiable Functions* , Punjab Univ. J. Math. Vol. **48**, No. 2 (2016) 35-48.

[12] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some ϕ-analogues of Hermite-Hadamard Inequality for s-convex Functions in the Second Sense and Related Estimates*, Punjab Univ. J. Math. Vol. **48**, No. 2 (2016) 147-166.

[13] M. Adil Khan, G. Ali Khan, M. Jameel, K. A. Khan and A. Kilicman, *New refinement of Jensen's Mercer inequality*, Journal of Computational and Theoretical Nanoscience **12**, (2015) 4442-4449.