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Modification of Homotopy Perturbation Algorithm Through Least Square Optimizer for Higher Order Integro-Differential Equations

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Abstract.: In this manuscript, modification of homotopy perturbation method (HPM) is proposed for integro-differential equations by coupling the least square method (LSM) with HPM. Improved accuracy in a very few iterations is the general advantage of this technique. The proposed method is applied to different higher order integro-differential equations of linear and nonlinear nature, and results are compared with exact as well as available solutions from the literature. Numerical and graphical analysis reveal that the proposed algorithm is reliable for integro-differential equations and hence can be utilized for more complex problems.

AMS (MOS) Subject Classification Codes: 3404; 65H20; 65L10; 35A25

Key Words: Homotopy Perturbation Method; Least square method; Boundary Value Problems; Integro-differential equations; Validity and convergence.

1. Introduction

Integro-differential equations (IDEs) arise in different fields such as diffusion process along atomic boundaries and epidemic modeling. Analytic solutions of such IDEs are not possible in most of the cases, and hence scientist use numerical or semi-numerical methods for analysis purposes. IDEs can also be seen in multi-phase processes. Examples of such processes are crystallization and chemical reactors operations [1]. Due to vast industrial applications, IDEs cannot be ignored. Their usage can be seen in many applied fields such as mathematical physics and engineering sciences. Morchalo investigated higher order IDEs of two point BVPs in [2]. Hamoud solved Fredholm IDEs numerically in [3]. Araghi proposed ADM solution of IDEs [4]. Haris and Manafian used modified Laplace decomposition method (MLDM) for IDEs [5]. In [6], Hamoud et al. analyzed nonlinear Voltera family IDEs through Laplace decomposition method. Various methods have been used for solutions of higher order IDEs and fractional IDEs [7, 8].

Recently, researchers are focusing on the development of more accurate and efficient schemes for DEs in initial, two point, multi-point and obstacle BVPs. These methods are of great interest for the scientific community to accurately explore and predict different situations. Few of these methods include Adomian decomposition method (ADM) and its different modifications [9], variation iterative method (VIM) [10], HPM and its various modifications [11, 12, 13, 14, 15], residual power series method (RPSM) [16], and Block-Pulse functions method with operational matrix [17].

In the current manuscript, HPM is combined with the method of least square, along with different homotopies and initial guesses to get accurate solutions of higher order IDEs. These changes enhance the accuracy of the obtained solutions as compared to other available schemes given in the literature.

2. Conceptual Framework of LSHPM for Integro-Differential Equations

Let us consider the following IDEs

$$\int_0^r \Phi(t)dt + N(\Phi) + L(\Phi) = f(r), \qquad r \in \Omega \tag{2. 1}$$

with

$$B(\Phi, \frac{d^n\Phi}{dr^n}) = 0 \qquad r \in \gamma. \tag{2. 2}$$

Here L, N, B are linear, nonlinear and boundary operators. $f(r)$ is known function while Φ is unknown function. Firstly, describe a homotopy $\zeta(q, r) : \Phi \times [0, 1] \to \mathbb{R}$, such that:

$$\Psi(\zeta, q) = (1-q)\left[L(\zeta) - L(\Phi_0)\right] + q\left[L(\zeta) + N(\zeta) + \int_0^r \zeta(t)dt - f(r)\right] = 0, \quad r \in \Omega \tag{2. 3}$$

$$\zeta(q, r) = \zeta_0 + \sum_{i=1}^{\infty} q^i \zeta_i, \tag{2. 4}$$

Place $q = 1$, Homotopy Perturbation solution of (2. 1) in series form is

$$\tilde{\Phi} = \lim_{q\to 1} \zeta(r, q) = \sum_{i=1}^{\infty} \zeta_i, \tag{2. 5}$$

To refine HPM solution, we reassign the dummy convergence controlling parameters $\alpha_i' s$ as coefficients in (2. 5), and substitute modified $\tilde{\Phi}$ in (2. 1) for obtaining the residual function as:

$$\mathbb{R}(r, \alpha_i) = L(\tilde{\Phi}(r, \alpha_i)) + N(\tilde{\Phi}(r, \alpha_i)) + \int_0^r \tilde{\Phi}(t)dt - f(r), \tag{2. 6}$$

Sum of square of residual (SSR) in this case is

$$\mathbb{J}(\alpha_i) = \int_I \mathbb{R}^2(r, \alpha_i)dx, \tag{2. 7}$$

In next step, we find the optimal values of $\alpha_i's$ from the system of equations $\frac{\partial J}{\partial \alpha_i} = 0, \quad \text{for} \quad i = 0, 1, 2, \cdots$, which significantly improve the accuracy of the obtained solution. For more details about the method can be seen in [18].

3. APPLICATION OF LSHPM TO INTEGRO-DIFFERENTIAL EQUATIONS

Problem 1. *Consider the fourth-order linear IDE* [19]*:*

$$\mathbb{G}^{(iv)}(x) - (1+e^x)x - 3e^x + \int_0^x \mathbb{G}(t)dt - \mathbb{G}(x) = 0, \qquad x \in (0,1) \tag{3. 8}$$

with

$$\mathbb{G}(0) = 1, \quad \mathbb{G}(1) = 1+e, \quad \mathbb{G}''(1) = 3e, \quad \mathbb{G}''(0) = 2.$$

with exact solution is $1 + xe^x$.

Solution: Firstly, we construct the following homotopy

$$(1-q)(\mathbb{G}^{iv}(x)) + q\left(\mathbb{G}^{iv}(x) - (1+e^x)x - 3e^x - \mathbb{G}(x) + \int_0^x \mathbb{G}(t)dt\right) = 0$$

which gives the following various order problems

Zeroth-order problem is

$$\mathbb{G}_0{}^{iv}(x) = 0, \quad \mathbb{G}_0(0) = 1, \quad \mathbb{G}_0''(0) = 2, \quad \mathbb{G}_0(1) = 1+e, \quad \mathbb{G}_0''(1) = 3e. \tag{3. 9}$$

The solution of (3. 9) is

$$\mathbb{G}_0(x) = \frac{1}{6}(6 - 4x + 3ex + 6x^2 - 2x^3 + 3ex^3). \tag{3. 10}$$

First order problem is

$$\begin{gathered} \mathbb{G}_1{}^{(4)}(x) + \int_0^x \mathbb{G}_0(t)\,dt - \mathbb{G}_0(x) - e^x x - x - 3e^x = 0, \\ \mathbb{G}_1(0) = 0, \quad \mathbb{G}_1''(0) = 0, \quad \mathbb{G}_1(1) = 0, \quad \mathbb{G}_1''(1) = 0. \end{gathered} \tag{3. 11}$$

The solution of (3. 11) is

$$\begin{aligned} \mathbb{G}_1(x) =& \frac{1}{120960}(x-1)(-9ex^7 + 6x^7 + 63ex^6 - 90x^6 - 21ex^5 + 358x^5 + \\ & 483ex^4 - 314x^4 + 483ex^3 + 4726x^3 - 41517ex^2 + 15422x^2 - \\ & 41517ex - 45058x + 120960e^x - 120960). \end{aligned} \tag{3. 12}$$

Suming up (3. 10) and (3. 12), first order HPM solution of (3. 8) is

$$\begin{aligned} \tilde{\mathbb{G}}(x) =& \frac{1}{6}\left(3ex^3 - 2x^3 + 6x^2 + 3ex - 4x + 6\right) + \frac{1}{120960}(x-1)(-9ex^7 + \\ & 6x^7 + 63ex^6 - 90x^6 - 21ex^5 + 358x^5 + 483ex^4 - 314x^4 + \\ & 483ex^3 + 4726x^3 - 41517ex^2 + 15422x^2 - 41517ex - 45058x + \\ & 120960e^x - 120960). \end{aligned} \tag{3. 13}$$

Since (3. 13) consists of $x^0, e^x, xe^x, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8$ and hence we have to find first order LSHPM solution of the form

$\tilde{\mathbb{G}}(x) = c_0 + c_1 e^x + c_2 x + c_3 e^x x + c_4 x^2 + c_5 x^3 + c_6 x^4 + c_7 x^5 + c_8 x^6 + c_9 x^7 + c_{10} x^8,$

Applying the boundary conditions from (3. 8), we obtain the values of c_0, c_1, c_2, c_3. After that replacing $\mathbb{G}$ with $\tilde{\mathbb{G}}$ in (3. 8) gives residual function

$$R(x, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}) = \tilde{\mathbb{G}}^{(iv)}(x) - x(1 + e^x) - 3e^x - \tilde{\mathbb{G}}(x) + \int_0^x \tilde{\mathbb{G}(t)} dt,$$

Next for finding optimal $c_i' s$, sum of squared residual is obtained as

$$\mathbb{J}(c_i) = \int_0^1 R^2(x, c_i) dx, \tag{3. 14}$$

Solving the system $\frac{\partial J}{\partial c_i} = 0$, leads to the following first order LSHPM solution

$$\tilde{\mathbb{G}}(x) = 1 + xe^x. \tag{3. 15}$$

Which is closed form solution. Results related to problem 1 are shown in Table 1.

	Exact	**LSHPM**		**HPM**	
x	**Sol.**	**Sol.**	**Error**	**Sol.**	**Error**
0.	1.	1.	0.	1.	0.
0.1	1.11052	1.11052	0.	1.11032	1.99×10^{-4}
0.2	1.24428	1.24428	0.	1.2439	3.76×10^{-4}
0.3	1.40496	1.40496	0.	1.40444	5.14×10^{-4}
0.4	1.59673	1.59673	0.	1.59613	5.97×10^{-4}
0.5	1.82436	1.82436	0.	1.82374	6.18×10^{-4}
0.6	2.09327	2.09327	0.	2.09269	5.78×10^{-4}
0.7	2.40963	2.40963	0.	2.40914	4.83×10^{-4}
0.8	2.78043	2.78043	0.	2.78009	3.45×10^{-4}
0.9	3.21364	3.21364	0.	3.21346	1.79×10^{-4}
1.	3.71828	3.71828	0.	3.71828	0.

Table 1: Comparison of first order solution and error in Problem 1.

Problem 2. *Fourth-order linear IDE* [7, 20]

$$\mathbb{G}^{(iv)}(x) = x(1 + e^x) + 3e^x + \mathbb{G}(x) - \int_0^x \mathbb{G}(t) dt, \qquad 0 < x < 1 \tag{3. 16}$$

$$\mathbb{G}(0) = 1, \quad \mathbb{G}(1) = 1 + e, \quad \mathbb{G}'(1) = 2e \quad \mathbb{G}'(0) = 1.$$

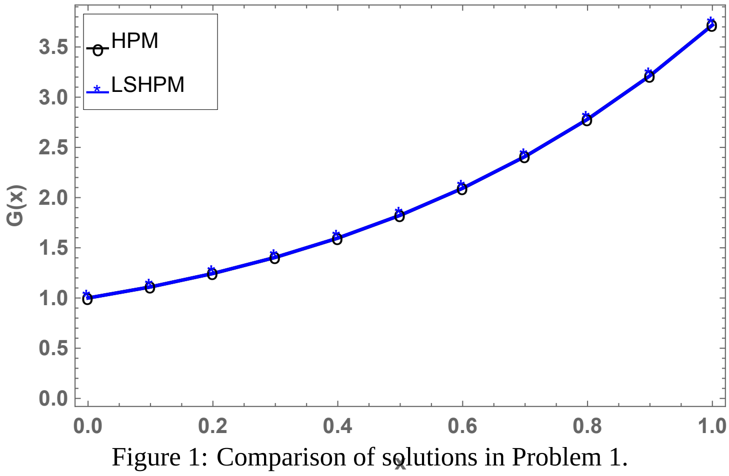

Figure 1: Comparison of solutions in Problem 1.

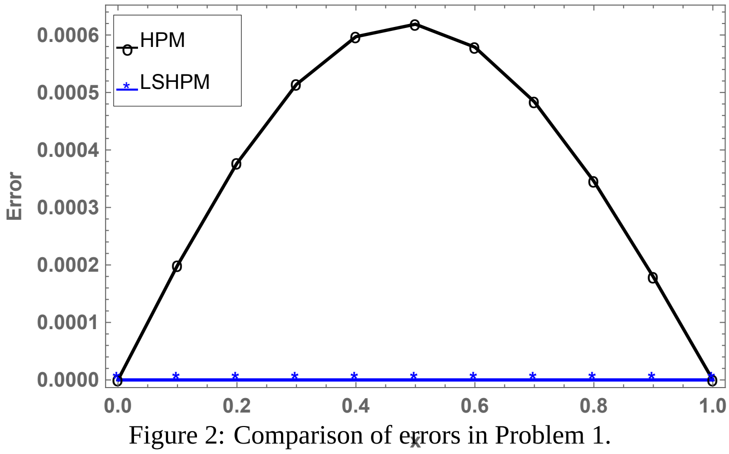

Figure 2: Comparison of errors in Problem 1.

Exact solution in this case is $1 + xe^x$.

Solution: After applying basic theory of LSHPM given in section 2, first order LSHPM solution is

$$\tilde{\mathbb{G}}(x) = 1 + xe^x.$$

which is closed form solution. Results related to this problem can be seen in Table 2.

Problem 3. *Fourth-order nonlinear IDE* [7, 20]

$$\mathbb{G}^{(iv)}(x) = 1 + \int_0^x e^{-t}\mathbb{G}^2(t)dt, \qquad 0 < x < 1 \tag{3. 17}$$

	Exact	LSHPM		HPM		VIM
x	Sol.	Sol.	Error	Sol.	Error	Error [?]
0.	1.	1.	0.	1.	0.	0.
0.1	1.11052	1.11052	0.	1.11051	3.40×10^{-6}	2.0×10^{-10}
0.2	1.24428	1.24428	0.	1.24427	1.10×10^{-5}	6.09×10^{-10}
0.3	1.40496	1.40496	0.	1.40494	1.92×10^{-5}	1.4×10^{-9}
0.4	1.59673	1.59673	0.	1.5967	2.50×10^{-5}	1.2×10^{-9}
0.5	1.82436	1.82436	0.	1.82433	2.66×10^{-5}	3.5×10^{-9}
0.6	2.09327	2.09327	0.	2.09325	2.36×10^{-5}	2.0×10^{-9}
0.7	2.40963	2.40963	0.	2.40961	1.71×10^{-5}	3.0×10^{-9}
0.8	2.78043	2.78043	0.	2.78042	9.27×10^{-6}	1.0×10^{-9}
0.9	3.21364	3.21364	0.	3.21364	2.67×10^{-6}	2.0×10^{-9}
1.	3.71828	3.71828	0.	3.71828	0.	1.0×10^{-9}

Table 2: Comparison of first order solution and error in Problem 2.

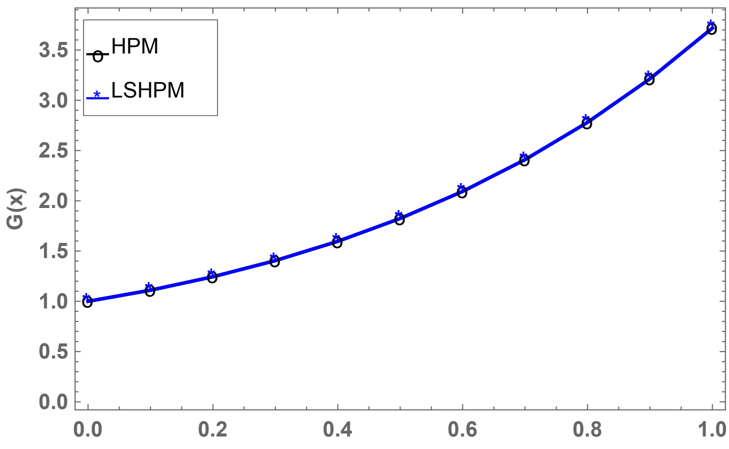

Figure 3: Comparison of solutions in Problem 2.

$$\mathbb{G}(1) = e, \quad \mathbb{G}(0) = 1, \quad \mathbb{G}'(1) = e, \quad \mathbb{G}'(0) = 1.$$

Exact solution in this case is e^x.

Solution: After applying proposed method which is given in section 2, zeroth-order LSHPM solution is

$$\tilde{\mathbb{G}}(x) = 7.77081 \times 10^{-17} + e^x + 7.77081 \times 10^{-17}x + 3.39245 \times 10^{-17}x^2 + 2.18918 \times 10^{-17}x^3$$

Results related to this problem are shown in table 3.

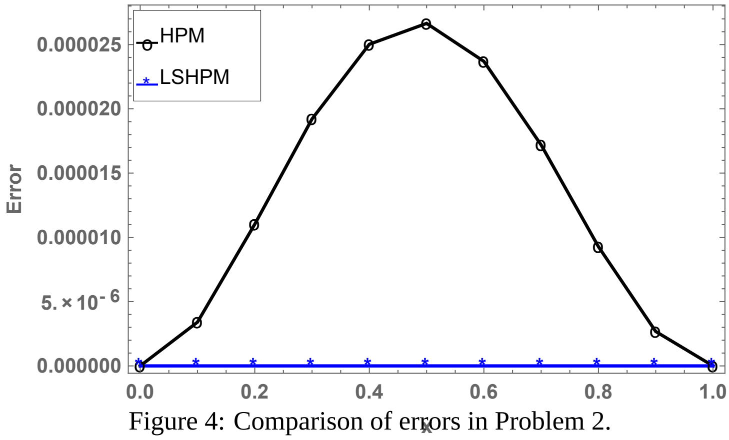

Figure 4: Comparison of errors in Problem 2.

x	Exact Sol.	LSHPM Sol.	LSHPM Error	HPM Sol.	HPM Error	VIM Error [?]
0.	1.	1.	0.	1.	0.	0.
0.1	1.10517	1.10517	0.	1.10412	1.04×10^{-3}	1.27×10^{-5}
0.2	1.2214	1.2214	0.	1.21803	3.37×10^{-3}	4.36×10^{-5}
0.3	1.34986	1.34986	0.	1.34394	5.92×10^{-3}	8.17×10^{-5}
0.4	1.49182	1.49182	2.22×10^{-16}	1.4839	7.88×10^{-3}	1.16×10^{-4}
0.5	1.64872	1.64872	2.22×10^{-16}	1.63999	8.73×10^{-3}	1.38×10^{-4}
0.6	1.82212	1.82212	2.22×10^{-16}	1.81391	8.20×10^{-3}	1.39×10^{-4}
0.7	2.01375	2.01375	0.	2.00734	6.41×10^{-3}	1.17×10^{-4}
0.8	2.22554	2.22554	0.	2.22174	3.80×10^{-3}	7.47×10^{-5}
0.9	2.4596	2.4596	0.	2.45838	1.22×10^{-3}	2.59×10^{-5}
1.	2.71828	2.71828	0.	2.71828	4.44×10^{-16}	0.

Table 3: Comparison of zeroth-order solution and error in Problem 3.

Problem 4. *Fifth order linear IDE* [21]

$$\mathbb{G}^{(v)}(x) = (e^x + 1)x + 4e^x + \mathbb{G}(x) - \int_0^x \mathbb{G}(t)dt, \qquad 0 < x < 3 \tag{3. 18}$$

with

$$\mathbb{G}(0) = 1, \quad \mathbb{G}(1) = 1 + e, \mathbb{G}'(0) = 1, \quad \mathbb{G}'(1) = 2e, \quad \mathbb{G}''(0) = 2.$$

Exact solution is $1 + xe^x$.

Solution: Use scheme given in section 2, first-order LSHPM solution is

$$\tilde{\mathbb{G}}(x) = 1 + xe^x.$$

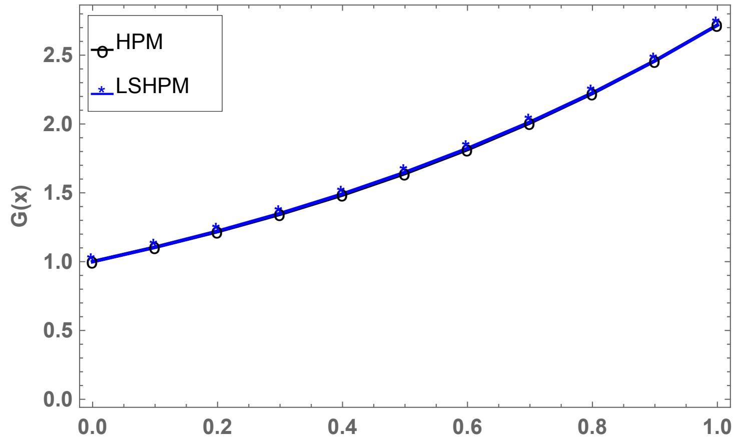

Figure 5: Comparison of solutions in Problem 3.

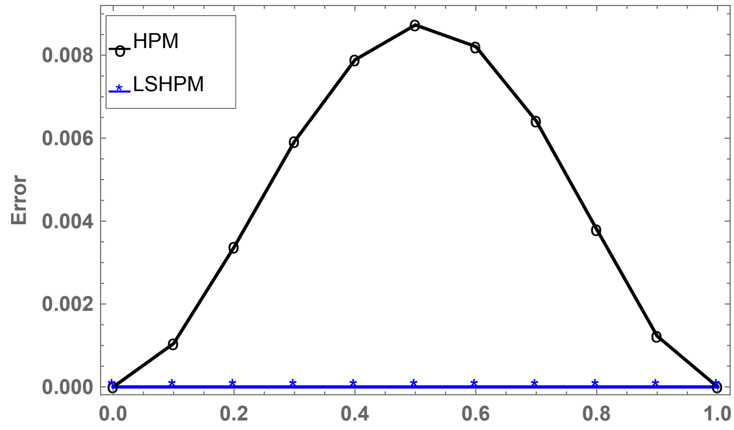

Figure 6: Comparison of errors in Problem 3.

Results related to this problem can be seen in table 4.

Problem 5. *Fifth order non-linear IDE* [21]

$$1+\int_0^x \mathbb{G}^3(t)e^{-2t}dx = \mathbb{G}^{(v)}(x), \qquad x \in (0,3) \tag{3. 19}$$

$$\mathbb{G}(3)=e^3, \quad \mathbb{G}(2)=e^2, \quad \mathbb{G}(1)=e, \quad \mathbb{G}'(0)=1, \quad \mathbb{G}(0)=1.$$

Exact solution in this case is e^x.

	Exact	LSHPM		HPM	
x	**Sol.**	**Sol.**	**Error**	**Sol.**	**Error**
0.	1.	1.	0.	1.	0.
0.1	1.11052	1.11052	0.	1.11052	6.45×10^{-9}
0.2	1.24428	1.24428	0.	1.24428	4.32×10^{-8}
0.3	1.40496	1.40496	0.	1.40496	1.18×10^{-7}
0.4	1.59673	1.59673	0.	1.59673	2.16×10^{-7}
0.5	1.82436	1.82436	0.	1.82436	3.06×10^{-7}
0.6	2.09327	2.09327	0.	2.09327	3.49×10^{-7}
0.7	2.40963	2.40963	0.	2.40963	3.19×10^{-7}
0.8	2.78043	2.78043	0.	2.78043	2.13×10^{-7}
0.9	3.21364	3.21364	0.	3.21364	7.57×10^{-8}
1.	3.71828	3.71828	0.	3.71828	4.44×10^{-16}

Table 4: Comparison of first order solution and error in Problem 4.

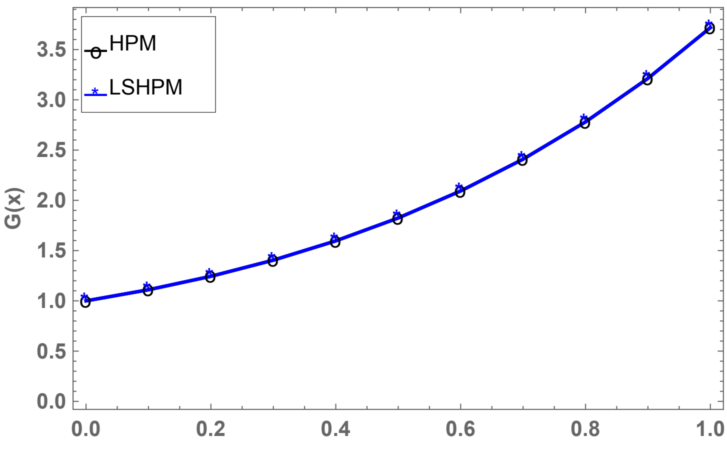

Figure 7: Comparison of solutions in Problem 4.

Solution: After applying general theory given in section 2, zeroth-order LSHPM solution is

$$\begin{aligned}\tilde{\mathbb{G}}(x) =& 1.0545 \times 10^{-15} + e^{x} + 1.0545 \times 10^{-15}x + 2.23652 \times 10^{-16}x^2 + 7.54882 \times 10^{-16}x^3 \\ &- 2.9813 \times 10^{-16}x^4 + 7.70205 \times 10^{-17}x^5\end{aligned}$$

Results related to this problem can be seen in Problem 5.

4. Conclusion

In this article, HPM and modified HPM were applied to higher order linear and nonlinear IDEs of order four and five. Validity of the obtained results is confirmed by comparing them

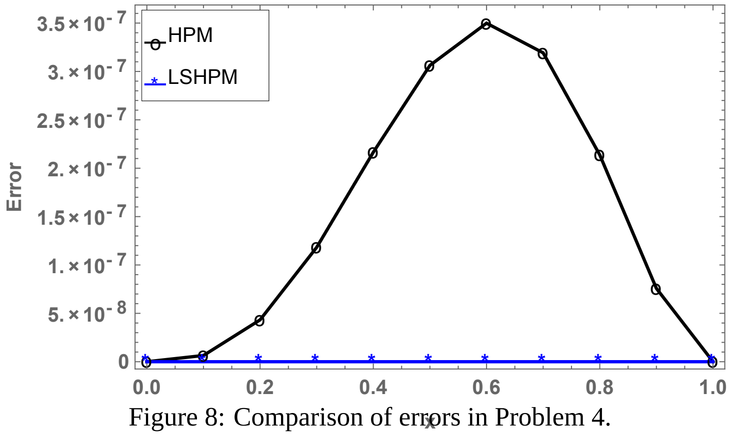

Figure 8: Comparison of errors in Problem 4.

x	Exact Sol.	LSHPM Sol.	LSHPM Error	HPM Sol.	HPM Error
0.	1.	1.	1.11×10^{-16}	1.	0.
0.1	1.10517	1.10517	2.22×10^{-16}	1.10728	2.10×10^{-3}
0.2	1.2214	1.2214	2.22×10^{-16}	1.22838	6.97×10^{-3}
0.3	1.34986	1.34986	2.22×10^{-16}	1.36258	1.27×10^{-2}
0.4	1.49182	1.49182	2.22×10^{-16}	1.50971	1.78×10^{-2}
0.5	1.64872	1.64872	2.22×10^{-16}	1.67009	2.13×10^{-2}
0.6	1.82212	1.82212	2.22×10^{-16}	1.84457	2.24×10^{-2}
0.7	2.01375	2.01375	0.	2.03452	2.07×10^{-2}
0.8	2.22554	2.22554	0.	2.2418	1.62×10^{-2}
0.9	2.4596	2.4596	0.	2.46877	9.16×10^{-3}
1.	2.71828	2.71828	0.	2.71828	0.

Table 5: Comparison of zeroth order solution and error in Problem 5.

with exact and other available solutions in the literature. Quantitative analysis has been performed in Tables [1-5] by presenting solutions along with errors in each case. These tables clearly indicate that LSHPM is reliable algorithm, and provide improved accuracy by incorporating few additional steps. Validity of proposed technique is also confirmed from graphical illustrations. It is clearly seen that modified method is an effective for IDEs, and can be extended to other families of differential equations.

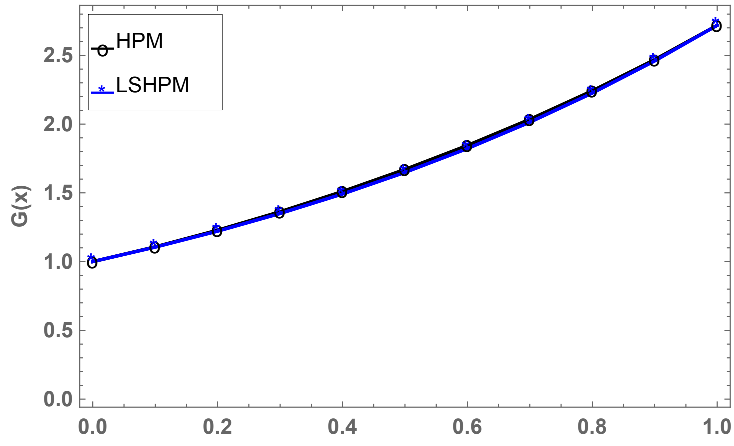

Figure 9: Comparison of solutions in Problem 5.

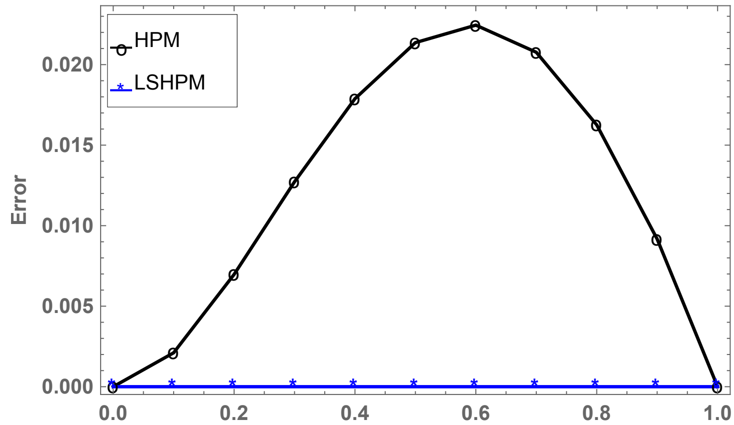

Figure 10: Comparison of errors in Problem 5.

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