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Numerical Study for Fractional-Order Magnetohydrodynamic Boundary Layer Fluid Flow Over Stretching Sheet

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Abstract.: In this letter, the MHD boundary layer fluid flow of non-Newtonian power-law on a stretching plate in the presence of a magnetic field has been investigated. The deductive group-theoretic technique is utilized to transform the proposed mathematical problem into a non-linear ODE. The solution of the converted differential equation is studied via the quartic B-spline method and the modified Laplace decomposition method. The approximate solutions are explained through tables and illustrative graphs for different values of the fractional order derivatives implementing the modified Laplace decomposition technique. We have used the Caputo sense of fractional derivative in this paper. A comparison of the obtained results reveals that both techniques are effective and reliable tools for the solutions of boundary value problems in fluid flow. It is found that when the pate and the fluid move in the same direction, the velocity profile declines and then improves at the end of the trend while the velocity profile gradually increases when the pate is stationary. The effect of the fractional order derivative on the velocity profile is another novelty of the present work. Furthermore, the influence of the physical parameters and the fractional order derivative on the stream function and the velocity profile is shown via tables and illustrative graphs.

AMS (MOS) Subject Classification Codes: 76A05;65N22;44A10:65Rxx:65L60.
Key Words: Magnetohydrodynamic, modified Laplace decomposition method, quartic B-spline method, magnetic field, velocity parameter.

1. Introduction

Various forms of differential equations of fractional and integer orders are appeared in modeling applied problems in optics, science, physics, ocean, finance, etc.[41, 26, 42, 13, 23, 37, 14, 40, 27, 10]. The vast majority of the fluids occurring in the manufacturing processes and applied sciences such as pulps, paper production, polymer etc. have rheological characteristics of boundary layer flow fluids. Because of the widely used of such fluid numerous efforts are directed to understand their friction and heat transfer properties. Sakiadis analyzed an incompressible Newtonian fluid boundary layer flow over an invariantly movable surface [38], and found out that these kinds of fluid problems lead to essentially various solutions compared to the flow problems investigated by Blasius, Latterly, and Pob. The exact solution for the same model is obtained by Crane [6]. The more comprehensive form of the heat transmission in a liquid layer over unsteady expansion sheet studied by Liu et al [18]. The second-order viscoelastic in nature fluids as a subclass of non-Newtonian fluids is examined via [8]. A stretching model of an incompressible fluid flow on porous wall is studied by Magyari and Keller [21]. The same authors investigated the heat transfer initiated via stretching surface in an incompressible fluid flow[5]. The heat and mass transmission in a viscous fluid electrically conducted on a stretching sheet with the existence of magnetic field is analyzed by Mishra et al. [24]. Bhatti et al. studied the effect of Magnetohydrodynamic on a stagnation point flow on a porous stretching plate using a numerical method [2]. Abel in [1] investigate heat transmission in liquid films with viscous properties with the existence of magnetic field. Rotational and hall current impacts over a convection magnetohydrodynamic flow is studied by [35]. The Magnetohydrodynamic Reiner-Rivlin flow via porous cylindrical annulus is analyzed by [7]. Very recently, the collocation method based on the quartic B-spline function and the well-known powerful technique Runge-Kutta method are employed to study the approximate solution of the Casson-Carreau fluid over a stretching sheet [28]. The collocation method based on spline function is utilized to investigate the numerical solution of the Casson fluid over a stretching sheet[29]. Over a continuously moving sheet, the influence of MHD fluid flow utilizing CNTs with thermal sink/source and radiation is studied in [22].

Several methods have been employed to investigate the solutions of differential equations and boundary value problems. The Laplace decomposition method is one of these methods which gives excellent solutions compare to the exact solutions see [11, 12]. The convergence of the Laplace decomposition method is studied in [4, 9]. Also, the error analysis of the collocation method based on quartic B-spline function is studied in [**?**].

The aim of the present work is to extend the analysis of non-Newtonian power-law boundary film flow fluid to a non-linear continuous moving surface by including the effect of the magnetic parameter. The Maple 2020 program is employed to solve the converted mathematical problem using the quartic B-spline technique and the modified Laplace decomposition method (MLDM). Numerical and graphical representations of the impact of emerging physical parameters on the skin friction, stream function, and fluid velocity are provided.

2. PRELIMINARIES

In this potion, some basic properties and definitions of Laplace transform and fractional calculus are introduced.

Definition 2.1. [39] *A real valued function* $g(y), y > 0$ *is belong to the space* $C_\sigma, \sigma \in R$ *if there exists* $d > \sigma$, *such that* $g(y) = y^d g_1(y)$ *where* $g_1(y) \in C(0, \infty)$, *and it is belong to the space* C_σ^n *if* $g^n \in R_\sigma$, $n \in N$.

Definition 2.2. [34] *The function* $f(x)$ *is called Riemann-Liouvill fractional integral if it defines as:*

$$D^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt,\ t > 0,$$

where $D^0 f(x) = f(x)$.

Some properties of the operator forD^α where $\mu \geq 0$ and $\tau \geq -1$:

(1) $D^\alpha D^\mu f(x) = D^{\alpha+\mu} f(x)$,
(2) $D^\alpha D^\mu f(x) = D^\mu D^\alpha f(x)$
(3) $D^\alpha y^\tau = \frac{\Gamma(\tau+1)}{\Gamma(\alpha+\tau+1)} y^{\alpha+\tau}$

Definition 2.3. [34] *The Caputo fractional derivative* $f \in C_{-1}^n$, $n \in N$ *is defined as :*

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^n(t)dt, n-1 < \alpha \leq n.$$

3. MATHEMATICAL MODEL

Consider an incompressible viscous power law fluid electrically conducting through a continuous moving sheet via a constant W_m in the opposite or same direction as the stream velocity W_∞. The x- and y- axes are parallel, and the x- axis grows parallel to the sheet. A magnetic field of uniform force B_0 connects the$y$- axis and produces a magnetic field in the x- axis orientation. The governing equations are

$$\begin{aligned} & w_x + v_y = 0, \\ & w w_x + v w_y = \frac{\tau_{xyy}}{\rho} - \frac{\sigma {B_0}^2 w}{\rho}, \end{aligned} \tag{3. 1}$$

where both w and v represent the velocity components in x and y orientations, respectively, ρrepresents the fluid density, τ_{xy} represents the shear stress associated to the boundary conditions:

$$\begin{aligned} & w = W_m,\ \ v = 0, \quad y = 0, \\ & w = W_\infty, \qquad as\, y \to \infty. \end{aligned} \tag{3. 2}$$

The relation of power law between shear rate and shear stress is implemented via

$$\tau_{yx} = \left|\frac{\partial w}{\partial y}\right|^{n-1} \frac{\partial w}{\partial y} K, \tag{3. 3}$$

where $\lambda \left|\frac{\partial w}{\partial y}\right|^{n-1}$ is the kinematic viscosity and the consistency coefficient represents by K, n is the index of power-law and $\lambda = \frac{K}{\rho}$. Substituting the kinematic viscosity value in

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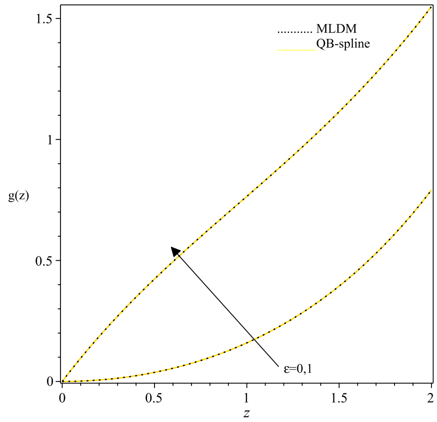

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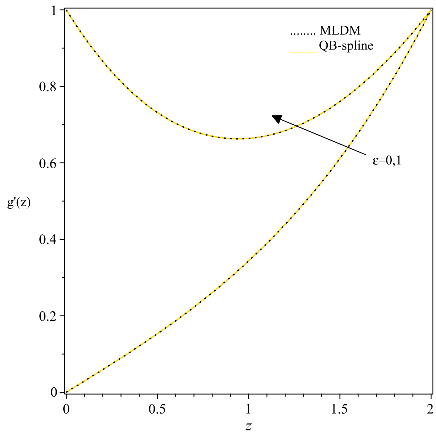

FIGURE 1. Comparison of $g'(z)$ and solution $g(z)$ of (6.31) for $\alpha = 3$ and various values of M.

equation (3.1), we obtain

$$ww_x + vw_y = \frac{\partial}{\partial y}\left(\lambda\left|\frac{\partial w}{\partial y}\right|^{n-1}\frac{\partial w}{\partial y}\right) - \frac{\sigma {B_0}^2 w}{\rho}, \tag{3. 4}$$

Here, we introduce the stream $\Omega(x, y)$ as $w = \frac{\partial\Omega}{\partial y}$, $v = \frac{\partial\Omega}{\partial x}$, substituting the velocity components w and v in equation (3.1), we get

$$\frac{\partial\Omega}{\partial y}.\frac{\partial^2\Omega}{\partial x\partial y} - \frac{\partial\Omega}{\partial x}.\frac{\partial^2\Omega}{\partial y^2} = \frac{\partial}{\partial y}\left(\lambda\left|\frac{\partial^2\Omega}{\partial y^2}\right|^{n-1}\frac{\partial^2\Omega}{\partial y^2}\right) - \frac{\sigma {B_0}^2}{\rho}\frac{\partial\Omega}{\partial y}, \tag{3. 5}$$

associate with the boundary conditions

$$\frac{\partial \Omega}{\partial y}(x,0) = W_m,\, \frac{\partial \Omega}{\partial x}(x,0) = 0,\; \frac{\partial \Omega}{\partial y}(x,\infty) = W_\infty, \tag{3. 6}$$

Via group-theoretic technique [3, 36] and considering the following transformations

$$\Omega(x,y) = ax^{\lambda} g(z),\; z = byx^{-\mu}, \tag{3. 7}$$

where $g(z)$ and z represent the dimensionless stream function and similarity variable respectively. $\lambda,\ \mu,\ a,$ and b are real numbers. By implementing the above transformations, we obtain the following

$$\begin{aligned} &\frac{\partial \Omega}{\partial x} = ax^{\lambda-1}[\lambda g - z\mu g'],\; \frac{\partial \Omega}{\partial y} = abg'\, x^{\lambda-\beta}, \\ &\frac{\partial^2 \Omega}{\partial y^2} = ab^2 g''\, x^{\lambda-2\beta}, \frac{\partial^2 \Omega}{\partial x \partial y} = abx^{\lambda-\mu-1}[\lambda g - \mu g' - z\beta g'']. \end{aligned} \tag{3. 8}$$

Inserting equation (3.7) and equation (3.8) into equation (3.5), the governing equations are converted to the following nonlinear differential equation:

$$\left(|g'|^{n-1} g''\right)' - Mg' - (\lambda-\mu){g'}^2 + \lambda g g'' = 0, \tag{3. 9}$$

if $\lambda(2-n) + \mu(2n-1) = 1$ and $a^{2n-1} b^{n-2} = 1$ holds, and the boundary conditions (3.6) gives $ab = W_\infty$ and $\lambda - \beta = 0$. Thus, inserting the above relations into equation (3.9), we obtain

$$\left(|g'|^{n-1} g''\right)' - Mg' + \frac{gg''}{n+1} = 0, \tag{3. 10}$$

associated with the boundary conditions

$$g(0) = 0,\, g'(0) = \varepsilon,\, g'(\infty) = 1.$$

The Newtonian fluid is obtained when $n = 1$. Thus, equation (3.10) becomes

$$g''' - Mg' + \frac{1}{2} g g'' = 0, \tag{3. 11}$$

associated with the boundary conditions

$$g'(0) = \varepsilon,\, g(0) = 0,\, g'(\infty) = 1, \tag{3. 12}$$

where $M = \frac{\sigma B_0^2}{\rho W_\infty}$ is the magnetic parameter and $\varepsilon = \frac{W_m}{W_\infty}$ is the parameter of velocity. Here, the fluid and plate move with the same velocity and direction when $\varepsilon = 1$, and the plate is stationary when $\varepsilon = 0$.

4. Quartic B-spline method

In boundary value problems and approximate theory, the B-spline function has a significant attention when numerical considerations are taken into account. In this portion, the B-spline function is introduced to derive an approximate solution to the suggested model. Starting with the finite set of points and dividing the interval $J = [a,b]$to$m$ sub intervals $J_i = [z_i, z_{i+1}]$ $(i = 0,1,2,...,m-1)$ by the nods $z_i = a + ih\,(i = 0,1,...,m)$, when $h = (b-a)/m$. The quartic spline space defined in [32] as follows:

$$S_4(J) = \left\{ s(z) \in C^3(J) | s(z)_{|J_i} \in P_4, i = 0,1,...,m-1 \right\},$$

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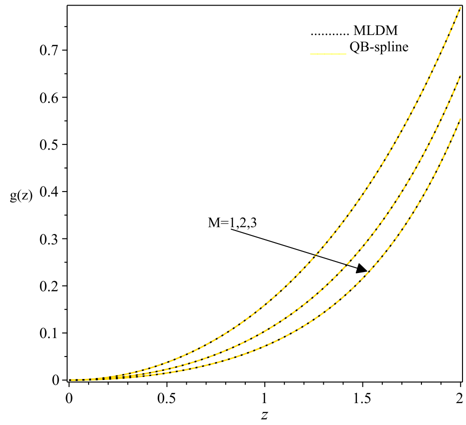

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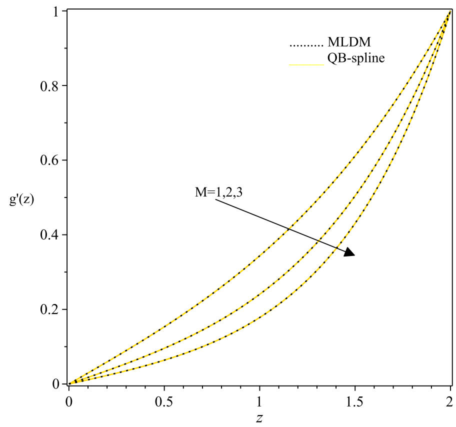

FIGURE 2. Comparison of $g(z)$ and $g'(z)$ of (3.11) and (3.12) for $M = 1$ and several values of ε

where $s(z)_{|J_i}$ is the strain of $s(z)$ on J_i, and the set of univariant quartic polynomials is denoted by P_4. Extending $J = [a, b]$ to $\tilde{J} = [a - 4, b + 4h]$ for the equal space nodes $z_i = a + ih\,(i = -4, -3, ..., m + 4)$. Now, the well-known B-spline [32, 33] is obtained as follows:

$$B_i(z) = \begin{cases} (z - z_{i-2})^4\,, & , I_1 \\ h^4 + 4h^3(z - z_{i-1}) + 6h^2(z - z_{i-1})^2 + 4h(z - z_{i-1})^3 - 4(z - z_{i-1})^4 & , I_2 \\ 11h^4 + 12h^3(z - z_i) - 6h^2(z - z_i)^2 - 12h(z - z_i)^3 + 6(z - z_i)^4 & , I_3 \\ h^4 + 4h^3(z_{i+2} - z) + 6h^2(z_{i+2} - z)^2 + 4h(z_{i+2} - z)^3 - 4(z_{i+2} - z)^4 & , I_4 \\ (z_{i+3} - z)^4 & , I_5 \\ 0, otherwise. \end{cases}$$

TABLE 1. Quartic B-splines table and at nodes z_j.

z	z_{j-2}	z_{j-1}	z_j	z_{j+1}	z_{j+2}	z_{j+3}
$B_j(z)$	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$	0
$hB'_j(z)$	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{11}{2}$	$\frac{-1}{6}$	0
$h^2B''_j(z)$	0	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$	0
$h^3B'''_j(z)$	0	1	-1	1	-1	0

were $I_1 = z_{i-2} \leq z \leq z_{i-1}, I_2 = z_{i-1} \leq z \leq z_i, I_3 = z_i \leq z \leq z_{i+1}, I_4 = z_{i+1} \leq z \leq z_{i+2}, I_5 = z_{i+2} \leq z \leq z_{i+3}$, and $B_i(z)$ denotes the locally supported non-negative over $[z_{i-2}, z_{i+3}]$. It is clear that $B_i(z) = B_{i+1}(z+h)\,(i = -2, -1, ..., m)$. In addition, $B_i(z)(i = -2, ..., 1+m)$ is known as basis spline of $S_4(J)$, since $B_i(z)$ is linearly independent. Thus, the approximate solution of equation (3.11) is represented via the quartic B-spline function:

$$S(z) = \sum_{i=-2}^{m+1} q_i B_i(z), \tag{4. 13}$$

where q_i denote the unknown coefficients known as control points.
Using series solution (4.13) and Table 1, we obtain

$$\begin{aligned} S(z) &= \sum_{i=-2}^{m+1} q_i B_i(z) = \frac{1}{24}(q_{i-2} + 11q_{i-1} + 11q_i + q_{i+1}), \\ S'(z) &= \sum_{i=-2}^{m+1} q_i B_i{}'(z) = \frac{1}{6h}(-q_{i-2} - 3q_{i-1} + 3q_i + q_{i+1}), \\ S''(z) &= \sum_{i=-2}^{m+1} q_i B_i{}''(z) = \frac{1}{2h^2}(q_{i-2} - q_{i-1} - q_i + q_{i+1}), \\ S'''(z) &= \sum_{i=-2}^{m+1} q_i B_i{}'''(z) = \frac{1}{h^3}(-q_{i-2} + 3q_{i-1} - 3q_i + q_{i+1}). \end{aligned} \tag{4. 14}$$

Here, the implement of the quartic B-spline technique on differential equation (3.11) with boundary conditions (3.12) is discussed. Suppose $S(z) = \sum_{i=-2}^{m+1} q_i B_i(z)$ is the numerical solution of equation (3.11) with boundary conditions (3.12). The suggested model is discretized at the nodes z_j, for $i = 0, 1, ..., m$, and utilizing (3.11), we get:

$$2\frac{q_{i-2} - 3q_{i-1} + 3q_i - q_{i+1}}{h^3} = \frac{q_{i-2} + 11q_{i-1} + 11q_i + q_{i+1}}{24}\left(\frac{q_{i-2} - q_{i-1} - q_i + q_{i+1}}{2h^2}\right) - 2M\left(\frac{-q_{i-2} - 3q_{i-1} + 3q_i + q_{i+1}}{6h}\right).$$

TABLE 2. Comparison of $g''(0)$ of equations (6.30) and (6.31) for several values of M.

M	$\varepsilon=0$			$\varepsilon=1$		
	$h=0.1$	$h=0.05$	$MLDM$	$h=0.1$	$h=0.05$	$MLDM$
0.0	0.5443565	0.5442929	0.5442717	0	0	0
0.5	0.3938071	0.3937535	0.3937357	-0.4324108	-0.4324921	-0.4325191
1.0	0.2944163	0.2943864	0.2943904	-0.7703428	-0.7702805	-0.7702599
1.5	0.2256477	0.2257284	0.2257553	-1.0453086	-1.0449893	-1.0449167
2.0	0.1764286	0.1765681	0.1766145	-1.2765798	-1.2760658	-1.2758527
3.0	0.1129475	0.1130759	0.1131484	-1.6537124	-1.6523887	-1.6519242

Now, we have $m+1$ non-linear equations:

$$\frac{-q_{i-2}+3q_{i-1}-3q_i+q_{i+1}}{h^3}=\theta_i(M,z_i,q_{i-2},q_{i-1},q_i,q_{i+1}),\ \ i=0,1,2,...,m.$$

Similarly, using the conditions at $z=a$ and $z=b$, the following equations are obtained:

$$\begin{aligned}&q_{-2}+11q_{-1}+11q_0+q_1=0,\\&-q_{-2}-3q_{-1}+3q_0+q_1=6h\varepsilon,\\&-q_{n-2}-3q_{n-1}+3q_n+q_{n+1}=6h.\end{aligned}$$

Here, a system of $m+4$ non-linear equations with $m+4$ unknowns $(q_{-2},q_{-1},...,q_{m+1})$ is obtained. The above non-linear system is solved to obtain the quartic B-spline solution $S(z)=\sum_{i=-2}^{m+1}q_iB_i(z)$.

5. MODIFIED LAPLACE DECOMPOSITION METHOD (MLDM)

$$D_z^{\alpha}g+d_1(z)g''+d_2(z)g'+d_3(z)g=H(z),\ \ 2<\alpha\leq 3 \tag{5. 15}$$

$$g(0)=e_1,g'(0)=e_2,g''(0)=e_3 \tag{5. 16}$$

Here, the Laplace decomposition method is applied [17] on (5.15)

$$s^{\alpha}L[g]-s^{\alpha-1}e_1-s^{\alpha-2}e_2-s^{\alpha-3}e_3+L[d_1(z)g''+d_2(z)g'+d_3(z)g]=L[H(z)] \tag{5. 17}$$

According to the differentiation property, we get

$$L[g]=\frac{e_1}{s}+\frac{e_2}{s^2}+\frac{e_3}{s^3}+\frac{1}{s^{\alpha}}L[H(z)]-\frac{1}{s^{\alpha}}L[d_1(z)g''+d_2(z)g'+d_3(z)g]. \tag{5. 18}$$

According to the Laplace decomposition method, the solution of equation (5.15) has the form

$$g=\sum_{k=0}^{\infty}g_k \tag{5. 19}$$

The non-linear term is decomposed as

$$H(z) = \sum_{k=0}^{\infty} A_k \tag{5. 20}$$

where A_k represents the Adomian polynomials [25, 31]of $g_0, g_1, g_2, ..., g_m$ that is evaluated by the following formula

$$A_k = \frac{1}{m!}\frac{d^k}{d\lambda^k}\left[N\left(\sum_{j=0}^{\infty}\lambda^j g_j\right)\right]_{\lambda=0}, \; k = 0, 1, 2, 3, ... \tag{5. 21}$$

Using (5.19) and (5.20) in (5.18), we obtain

$$L[\sum_{k=0}^{\infty} g_k] = \frac{e_1}{s}+\frac{e_2}{s^2}+\frac{e_3}{s^3}+\frac{1}{s^\alpha}L[\sum_{k=0}^{\infty} A_k]-\frac{1}{s^\alpha}L[d_1(z)\sum_{k=0}^{\infty} g''_k+d_2(z)\sum_{k=0}^{\infty} g'_k+d_3(z)\sum_{k=0}^{\infty} g_k]. \tag{5. 22}$$

Matching both sides of (5.22), the following relations are acquired

$$L[g_0] = \frac{e_1}{s} + \frac{e_2}{s^2} + \frac{e_3}{s^3}, \tag{5. 23}$$

$$L[g_1] = \frac{1}{s^\alpha}L[A_0] - \frac{1}{s^\alpha}L[d_1(z)g''_0 + d_2(z)g'_0 + d_3(z)g_0],$$
$$L[g_2] = \frac{1}{s^\alpha}L[A_1] - \frac{1}{s^\alpha}L[d_1(z)g''_0 + d_2(z)g'_0 + d_3(z)g_0]. \tag{5. 24}$$

Generally, the recursive relation has the form

$$L[g_{k+1}] = \frac{1}{s^\alpha}L[A_k] - \frac{1}{s^\alpha}L[d_1(z)g''_k + d_2(z)g'_k + d_3(z)g_k], \; k \geq 0. \tag{5. 25}$$

Taking the inverse Laplace transform on equations (4.23)-(4.25), we obtain

$$g_0(z) = H(z),$$
$$g_{k+1}(z) = L^{-1}\left[\frac{1}{s^\alpha}L[A_k] - \frac{1}{s^\alpha}L[d_1(z)g''_k + d_2(z)g'_0 + d_3(z)g_k]\right], \; k \geq 0.$$

Here, the initial solution plays a crucial role in solving the proposed problem. Instead of the iteration procedure, the above initial condition is decomposed into two parts, $H_1(z)$ and $H_0(z)$ see [15]. Therefore, the following modifications are suggested

$$g_0(z) = H_0(z),$$
$$g_1(y) = H_1(z) + L^{-1}\left[\frac{1}{s^\alpha}L[A_0] - \frac{1}{s^\alpha}L[d_1(z)g''_0 + d_2(z)g'_0 + d_3(z)g_0]\right],$$
$$g_{k+1}(z) = L^{-1}\left[\frac{1}{s^\alpha}L[A_k] - \frac{1}{s^\alpha}L[d_1(z)g''_k + d_2(z)g'_k + d_3(z)g_k]\right], \; k \geq 1. \tag{5. 26}$$

Now, we aim in this paper to present our problem arising in fluid flow in fractional order such as in [19, 20, 30]. The fractional derivative of the converted mathematical model (3.11) is explained by the following fractional (ODE):

$$D_z^\alpha g - Mg' + \frac{1}{2}gg'' = 0,$$
$$g'(0) = \varepsilon, g(0) = 0, g''(0) = \lambda, \tag{5. 27}$$

where D_z^α represents the Caputo derivative and is expressed as[15, 16]:

$$D_z^\alpha g(\eta) = \frac{1}{\Gamma(l-\alpha)} \int_0^\eta (\eta-\zeta)^{l-\alpha-1} g^{(l)}(\zeta) d\zeta,\ l-1 < \alpha \le l,$$

where l is an integer number. The Caputo fractional derivative of Laplace transform expressed as:

$$L[D_z^\alpha g(\eta)] = s^\alpha L[g(\eta)] - \sum_{k=0}^{l-1} s^{\alpha-k-1} g^{(k)}(0).$$

Here, we apply the Laplace transform on non-linear fractional differential equations (5.27) as follows:

$$L[D_z^\alpha g] = -L\left[-Mg' + \frac{1}{2}gg''\right]$$

By the Caputo derivative of the Laplace transform, we obtain

$$L[g] = \frac{\varepsilon}{s^2} + \frac{\lambda}{s^3} - \frac{1}{s^\alpha} L\left[\frac{1}{2}gg'' - M\,g'\right] \tag{5. 28}$$

where $g''(0) = \lambda$. Implementing the inverse Laplace transform on (5.28) gives

$$g(z) = \varepsilon z + \frac{\lambda z^2}{2} - L^{-1}\left[\frac{1}{s^\alpha} L\left[\frac{1}{2}gg'' - M\,g'\right]\right]$$

The series solution of (4.27) is as follows:

$$g_0 = \varepsilon z + \frac{\lambda z^2}{2}$$

$$\sum_{k=0}^{\infty} g_k(z) = \varepsilon z + \frac{\lambda z^2}{2} - L^{-1}\left[\frac{1}{s^\alpha} L\left[\left(\frac{1}{2}\sum_{k=0}^{\infty} A_k - M\sum_{k=0}^{\infty} g'_k\right)\right]\right] \tag{5. 29}$$

The Adomian polynomials A_k, $k = 1, 2, 3$ represent the non-linear terms as follows:

$$A_0(z) = g''_0(z)g_0(z),\ A_1(z) = g''_0(z)g_1(z) + g''_1(z)g_0(z),$$
$$A_2(z) = g_0(z)g''_1(z) + g_1(z)g''_0(z) + g_1(z)g''_1(z).$$

Using the proposed technique, the function $H(z)$ can be written as

$$H(z) = g_0(z) + g_1(\eta) = \varepsilon z + \frac{\lambda z^2}{2}.$$

Here, we obtain the following recursive relation

$$g_0(z) = z\varepsilon,$$
$$g_1(z) = 0.5\lambda z^2 - L^{-1}\left[\frac{1}{s^\alpha} L\left[0.5A_0 - Mg'_0\right]\right],$$
$$g_{k+1}(z) = -L^{-1}\left[\frac{1}{s^\alpha} L\left[0.5A_k - Mg'_k\right]\right],\ k \ge 1.$$

Hence, the series solution is

$$g_0(z) = \varepsilon z,$$
$$g_1(z) = \frac{\lambda z^2}{2} + \frac{\varepsilon M\, z^\alpha}{\Gamma(\alpha+1)},$$

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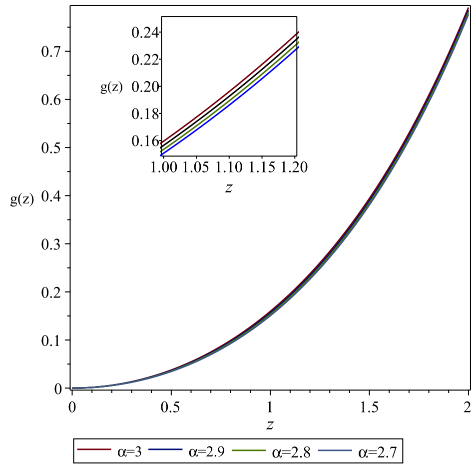

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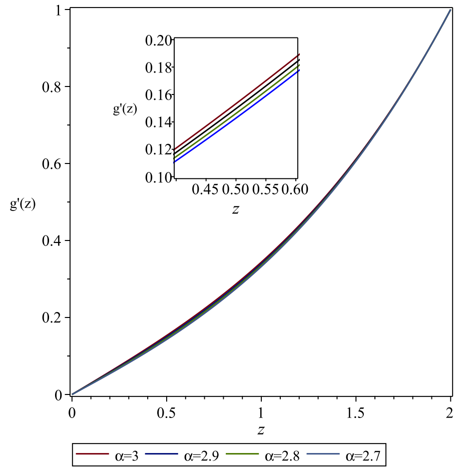

FIGURE 3. The effect of α on $g(z)$ and $g'(z)$ of (6.31) using MLDM when $M = 1$.

$$g_{k+1}(z) = -L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\frac{1}{2}A_k - M{g'}_k\right]\right],\ k \geq 1.$$

The value of $g''(0) = \lambda$ is found by determining $z_\infty = 2$ and using the condition $g'(\infty) = 1$.

6. RESULTS AND DISCUSSION

The numerical solutions that have been calculated using QB-spline and MLDM are discussed in this section. The solution of the proposed problem is investigated to show the

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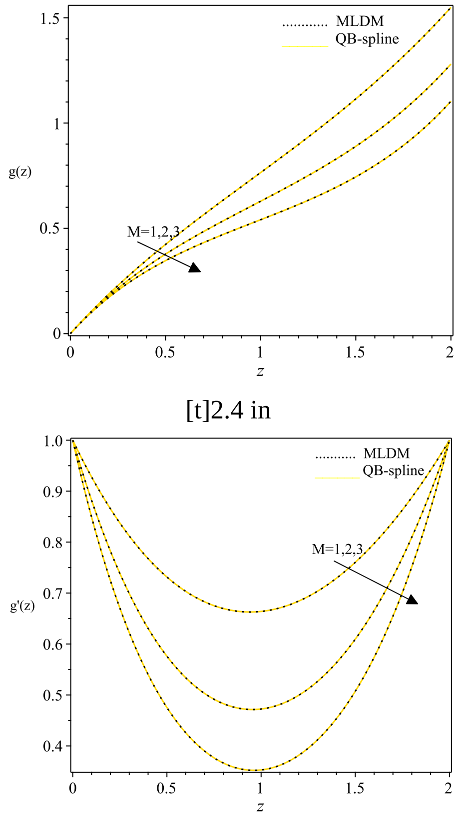

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FIGURE 4. Comparison of solution $g(z)$ and $g'(z)$ of equation (6.30) for $\alpha = 3$ and different values of M

TABLE 3. Comparison of values $-g''(0)$ of equations (3.11) and (3.12) for various values of M.

ε	$QB - spline$ $h = 0.1$	$h = 0.05$	$MLDM$
0.8	0.545069	0.544975	0.544995
0.6	0.325964	0.325829	0.325863
0.4	0.112983	0.112864	0.112894

behavior of unsteady flow for various magnetic field and velocity parameter values. As seen below, two physical impairments emerging in natural fluid flows are thoroughly investigated.

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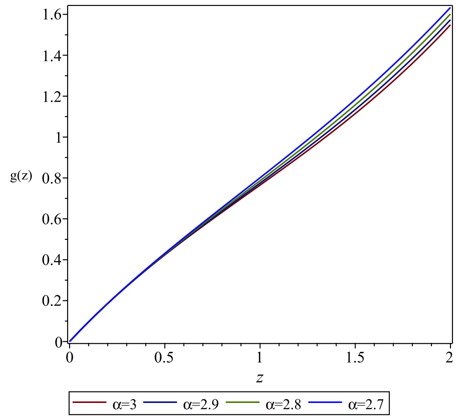

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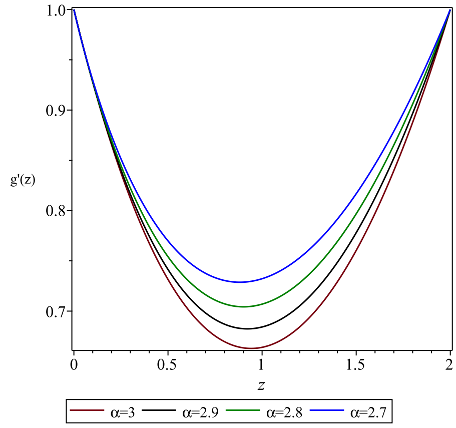

FIGURE 5. The impact of α on $g'(z)$ and $g(z)$ of equation (6.30) using MLDM when $M = 1$.

6.1. **The plate is stationary $\varepsilon = 0$.**

In the absence of velocity parameter ε, the plate is stationary and the following two-dimensional laminar flow is gained:

$$
\begin{gathered}
g''' + \frac{1}{2} g g'' - M g' = 0, \\
g(0) = 0, \quad g'(0) = 0, \quad g'(\infty) = 1.
\end{gathered} \tag{6. 30}
$$

The numerical solution of equation (6.30) using QB-spline method is founded as follows for $M = 1$:

$$g(z) = \begin{cases} 0.1471932031z^2 + 0.012225989z^4, & 0 \leq z \leq 0.05, \\ 0.1471922926z^2 + 0.000012124z^3 + 0.0121653z^4, & 0.05 \leq z \leq 0.1, \\ \vdots \\ -0.0423 + 0.09244z + 0.0737382z^2 + 0.023669z^3 + 0.01028z^4, & 1.95 \leq z \leq 1.95, \\ -0.0561 + 0.1209379z + 0.0517319z^2 + 0.031165z^3 + 0.00932z^4, & 1.95 \leq z \leq 2, \\ 0 & , otherwise. \end{cases}$$

The comparison of velocity profiles $g'(z)$ and solutions $g(z)$ when the plate is stationary $\varepsilon = 0$ and the plate and fluid move in the same direction $\varepsilon = 1$ are shown in Figure 1., the displacement of fluid when $\varepsilon = 0$ is less than the displacement of fluid when $\varepsilon = 1$. Figure 2. illustrates the comparison of velocity profile $g'(z)$ and solution $g(z)$ of (4.27) utilizing both proposed methods and demonstrates the influence of magnetic parameter M on the velocity profile $g'(z)$ and solution $g(z)$. In Figure 2. an increment of the value of M reduces the solution and velocity profile. Figure 3. shows the impact of α on $g(z)$ and $g'(z)$. Decreasing the value of α slightly decays the value of $g(z)$ and decreasing the value of α has no remarkable effect on the value of velocity profile $g'(z)$. In addition, Table 2. demonstrates the comparison of values of wall shear stress $g''(0)$ acquired from equations (4.27) and (4.28) with those obtained by QB-spline and MLDM for various chosen values of magnetic parameter M. An excellent agreement can be observed between the results presented numerically and graphically.

6.2. **The plate and fluid move same direction $\varepsilon = 1$.**

With the presence of the velocity parameter, the following boundary layer flow is obtained:

$$\begin{aligned} &g''' - Mg' + gg'' = 0, \\ &g'(0) = 1, \; g(0) = 0, \quad g'(\infty) = 0. \end{aligned} \tag{6. 31}$$

The numerical solution of equation (5.31) using QB-spline method is founded as follows for $M = 1$:

$$g(z) = \begin{cases} z - 0.385140287z^2 + 0.1666666z^3 - 0.016327z^4, & 0 \leq z \leq 0.05, \\ z - 0.385147z^2 + 0.1667563z^3 - 0.016775z^4, & 0.05 \leq z \leq 0.1, \\ \vdots \\ 0.0172 + 0.94238z - 0.3088482z^2 + 0.117845z^3 - 0.00381z^4, & 1.90 \leq z \leq 1.95, \\ 0.0136 + 0.949953z - 0.3145905z^2 + 0.119791z^3 - 0.00406z^4, & 1.95 \leq z \leq 2, \\ 0, & otherwise. \end{cases}$$

The influence of the magnetic parameter on the solution $g(z)$ and $g'(z)$ of (4.28) is demonstrated in Figure 4. In Figure 4, the increment of M decreases the value of solution and velocity profile. However, increasing the value of α in Figure 5. clearly, results a rising the value of solution $g(z)$ and velocity profile $g'(z)$. In Table 3, the effect of the value of velocity parameter ε on the wall shear stress $-g''(0)$ is presented. It can be seen from Table 3. that the value of $-g''(0)$ improves with increment in the value of velocity parameter ε.

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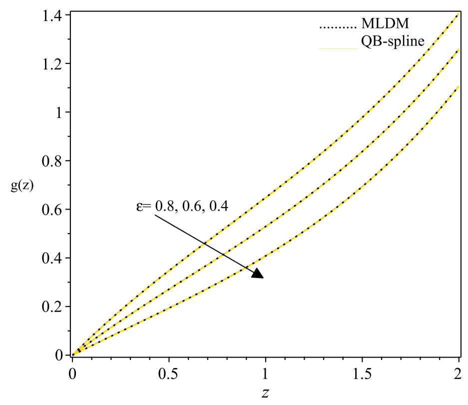

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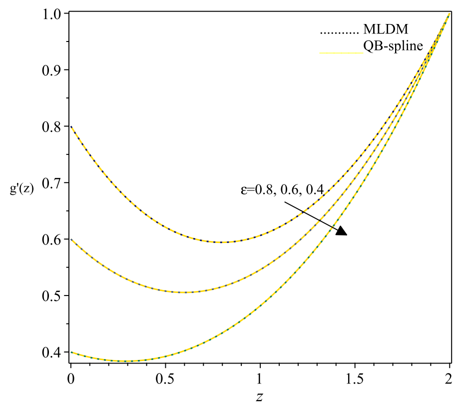

FIGURE 6. Comparison of $g'(z)$ and solution $g(z)$ of (3.11) and (3.12) for $M = 1$ and $\alpha = 3$

Further, there is a remarkable consistency between the solutions utilizing QB-spline and MLDM for all values of velocity parameter.

7. CONCLUSION

In this paper, we study the influence of the velocity and magnetic field parameters on the boundary flow over continuous moving sheet for non-Newtonian power-law fluid. The deductive group-theoretic technique is utilized to transform the proposed mathematical problem into a non-linear ODE and investigated its solution approximately utilizing the quartic B-spline technique. The suggested model is converted to a model of fractional flow fluid utilizing Caputo derivative that dissect the influence of the memory of characteristic of flow fluid and solved via implementing a method of Laplace decomposition transform. We have studied the influence of the fractional order on the stream function and . The comparison of the obtained results shows the accuracy of the suggested techniques. Further, the impact of magnetic parameter and velocity parameter on and the fluid velocity is discussed. Finally, we conclude from tables and illustrative graphs that B-spline functions and Laplace decomposition method are good tools and efficient techniques for solving the fractional boundary layer models. In the future, one can study the solutions of the present model for and that provide a rich investigation for further study. Here, the question arises whether these solutions are physically realizable or not.

REFERENCES

[1] M. S. Abel, J. Tawade, and M. M. Nandeppanavar, *Effect of non-uniform heat source on MHD heat transfer in a liquid film over an unsteady stretching sheet,* Int. J. Non. Linear. Mech., **44**(2009)990998.

[2] M. M. Bhatti, M. A. Abbas, and M. M. Rashidi, *A robust numerical method for solving stagnation point flow over a permeable shrinking sheet under the influence of MHD,* Appl. Math. Comput., **316**(2018)381389.

[3] G. Brikhoff, *Hydrodynamic.* Princenton Univ. Press, New York, 1950.

[4] Y. Cherruault and G. Adomian, *Decomposition methods: a new proof of convergence,* Math. Comput. Model., **18**(1993)103106.

[5] R. Cortell, *Similarity solutions for flow and heat transfer of a viscoelastic fluid over a stretching sheet,* Int. J. Non. Linear. Mech., **29**(1994)155161.

[6] L. J. Crane, *Flow past a stretching plate,* Zeitschrift fr Angew. Math. und Phys. ZAMP, **21**(1970)645647.

[7] S. Deo and S. Kumar, *MHD Reiner-Rivlin Liquid Flow Through a Porous Cylindrical Annulus.,* Appl. Appl. Math., **16**(2021)1038-1056.

[8] S. A. Gaffar, O. A. Bg, and V. R. Prasad,*Mathematical modeling of natural convection in a third-grade viscoelastic micropolar fluid from an isothermal inverted cone,* Iran. J. Sci. Technol. Trans. Mech. Eng., **44** (2020)383402.

[9] M. M. Hosseini and H. Nasabzadeh, *On the convergence of Adomian decomposition method,* Appl. Math. Comput., **182**(2006)536543.

[10] M. Huang, M. A. S. Murad, O. A. Ilhan, and J. Manafian, *One-, two-and three-soliton, periodic and cross-kink solutions to the (2+ 1)-D variable-coefficient KP equation,* Mod. Phys. Lett. B, **34**(2020) 2050045.

[11] O. Gonzlez-Gaxiola, A. Biswas, M. Asma, and A. K. Alzahrani, *Highly dispersive optical solitons with non-local law of refractive index by Laplace-Adomian decomposition,* Opt. Quantum Electron., **53**,(2021)112.

[12] S. Gupta, D. Kumar, and J. Singh, *Analytical solutions of convectiondiffusion problems by combining Laplace transform method and homotopy perturbation method,* Alexandria Eng. J., **54**(2015)645651.

[13] H. F. Ismael, M. A. S. Murad, and H. Bulut, *M-lump waves and their interaction with multi-soliton solutions for a generalized KadomtsevPetviashvili equation in (3+ 1)-dimensions,* Chinese J. Phys.,**77**(2022) 13571364.

[14] H. F. Ismael, M. A. S. Murad, and H. Bulut, *Various exact wave solutions for KdV equation with time-variable coefficients,* J. Ocean Eng. Sci., **7** (2022) 409418.

[15] Y. Khan, *An effective modification of the Laplace decomposition method for nonlinear equations,* Int. J. Nonlinear Sci. Numer. Simul., **10**(2009)13731376.

[16] Y. Khan and N. Faraz, *Application of modified Laplace decomposition method for solving boundary layer equation,* J. King Saud Univ. Sci., **23**(2011)115119.

[17] S. A. Khuri, *A Laplace decomposition algorithm applied to a class of nonlinear differential equations,* J. Appl. Math., **1**(2001)141155.

[18] I. C. Liu and H. I. Andersson, *Heat transfer in a liquid film on an unsteady stretching sheet,* Int. J. Therm. Sci., **47**(2008)766772.

[19] L. A. Lund, Z. Omar, S. O. Alharbi, I. Khan, and K. S. Nisar, *Numerical investigation of multiple solutions for caputo fractional-order-two dimensional magnetohydrodynamic unsteady flow of generalized viscous fluid over a shrinking sheet using the Adams-type predictor-corrector method,* Coatings, **9**(2019)548.

[20] K. R. Madhura, B. Atiwali, and S. S. Iyengar, *Influence of nanoparticle shapes on natural convection flow with heat and mass transfer rates of nanofluids with fractional derivative,* Math. Methods Appl. Sci., 2021.

[21] E. Magyari and B. Keller, *Exact solutions for self-similar boundary-layer flows induced by permeable stretching walls,* Eur. J. Mech., **19**,(2000)109122.

[22] U. S. Mahabaleshwar, K. N. Sneha, A. Chan, and D. Zeidan, *An effect of MHD fluid flow heat transfer using CNTs with thermal radiation and heat source/sink across a stretching/shrinking sheet,* Int. Commun. Heat Mass Transf., **135**(2022)106080.

[23] J. Manafian, M. A. S. Murad, A. Alizadeh, and S. Jafarmadar, *M-lump, interaction between lumps and stripe solitons solutions to the (2+ 1)-dimensional KP-BBM equation,* Eur. Phys. J. Plus, **135**(2020)120.

[24] S. R. Mishra, P. K. Pattnaik, M. M. Bhatti, and T. Abbas, *Analysis of heat and mass transfer with MHD and chemical reaction effects on viscoelastic fluid over a stretching sheet,* Indian J. Phys., **91**(2017)12191227.

[25] D. H. Malo, M. A. S. Rogash Younis Masiha, and S. T. A. Murad, *A New Computational Method Based on Integral Transform for Solving Linear and Nonlinear Fractional Systems,* J. Mat. MANTIK, **7**,(2021)919.

[26] M. A. S. Murad, F. K. Hamasalh, and H. F. Ismael, *Various optical solutions for timefractional Fokas system arises in monomode optical fibers,* Opt. Quantum Electron., **55**(2023)300.

[27] M. A. S. Murad, *Property Claim Services by Compound Poisson Process And Inhomogeneous Levy Process,* Sci. J. Univ. Zakho, **6**(2018)3234.

[28] M. A. S. Murad, F. K. Hamasalh, and H. F. Ismael, *Numerical study of stagnation point flow of Casson-Carreau fluid over a continuous moving sheet,* AIMS., **8**(2023)70057020.

[29] M. A. S. Murad, F. K. Hamasalh, and H. F. Ismael, *Numerical study of stagnation point flow of Casson fluid over a continuous moving sheet,* Frontiers in Heat and Mass Transfer (FHMT).

[30] M. A. S. Murad, *Modified integral equation combined with the decomposition method for time fractional differential equations with variable coefficients,* Appl. Math. J. Chinese Univ., **37**(2022)404414.

[31] M. A. S. Murad and M. H. Hamadamen, *A New analytical Modeling for Fractional Telegraph Equation Arising in Electromagnetic,* J. Mat. MANTIK, **8**(2022)124138.

[32] M. A. S. Murad and F. K. Hamasalh, *Computational Technique for the Modeling on MHD Boundary Layer Flow Unsteady Stretching Sheet by B-Spline Function,* in 2022 International Conference on Computer Science and Software Engineering (CSASE),(2022)236240.

[33] K. Parand and N. Bajalan, *A numerical approach based on B-spline basis functions to solve boundary layer flow model of a non-Newtonian fluid,* J. Brazilian Soc. Mech. Sci. Eng., **40**(2018)19.

[34] A. Prakash and V. Verma, *Numerical method for fractional model of newell-whitehead-segel equation,* Front. Phys., **7**(2019) 110.

[35] U. S. Rajput and N. K. Gupta, *(R1501) Rotational and Hall Current Effects on a Free Convection MHD Flow with Radiation and Inclined Magnetic Field,* Appl. Appl. Math. An Int. J., **17**(2022)10. , **20**(2023)1-8.

[36] R. Seshadri and T. Y. Na, *Group invariance in engineering boundary value problems.* Springer Science and Business Media, 2012.

[37] P. Satapathy, T. Raja Sekhar, and D. Zeidan, *Codimension two Lie invariant solutions of the modified KhokhlovZabolotskayaKuznetsov equation,* Math. Methods Appl. Sci., **44**(2021)49384951.

[38] B. C. Sakiadis, *Boundarylayer Behavior on Continuous Solid Surfaces: II. The Boundary Layer on a Continuous Flat Surface,*AiChE Journal, **7**(1961)22-25.

[39] R. V Slonevskii and R. R. Stolyarchuk, *Rational-fractional methods for solving stiff systems of differential equations,* J. Math. Sci., **150**(2008)24342438.

[40] F. Sultana, D. Singh, R. K. Pandey, and D. Zeidan, *Numerical schemes for a class of tempered fractional integro-differential equations,* Appl. Numer. Math., **157**(2020)110134.

[41] D. Zeidan, C. K. Chau, and T. Lu, *On the characteristic Adomian decomposition method for the Riemann problem,* Math. Methods Appl. Sci., **44**(2021)80978112.

[42] D. Zeidan, C. K. Chau, T. Lu, and W. Zheng, *Mathematical studies of the solution of Burgers equations by Adomian decomposition method,* Math. Methods Appl. Sci., **43**(2020)21712188.