Punjab University Journal of Mathematics (2023),55(5-6),185-196
https://doi.org/10.52280/pujm.2023.55(5-6)01

# On the Fractional Integral Inequalities for $p$-convex Functions

Hanlar MAMEDOV(RESIDOGLU)
Department of Mathematics,
University of Igdir, Igdir, Turkey,
Email: hanlar@mersin.edu.tr

Ilknur YESILCE ISIK
Department of Mathematics,
University of Aksaray, Aksaray, Turkey,
Email: ilknuryesilce@gmail.com

Received: 29 November, 2022 / Accepted: 29 May, 2023 / Published online:30 May,2023

**Abstract.**: Inequality theory is one of the major application areas of convexity. Nowadays, the introduction of novel generalizations of convexity has yielded considerable contributions in inequality theory. In this article, some applications of $p$-convexity via fractional calculus to this field is studied. The inequalities for Riemann-Liouville fractional integrals of p-convex functions are obtained. Also, the special cases of these results are presented and exemplified for some of $p$-convex functions.

**AMS (MOS) Subject Classification Codes: 26B25; 52A40; 91B52**
**Key Words:** $p$-convexity, Hermite-Hadamard Inequality, Fractional Integral, Riemann-Liouville.

## 1. Introduction

The convexity has numerous applications to the inequality theory. Also, the inequalities stated for convex functions can be used to obtain other inequalities, for instance, Minkowskis triangle inequality and Holders inequality are obtained by applying Jensen's inequality for certain convex functions [1]. Hermite-Hadamard inequality is one of the celebrated inequalities for convex functions, which establishes the relations of various means associated to the function [20, 21]. If the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function, then the following inequality holds

$$f\left(\frac{\mu+\nu}{2}\right) \leq \frac{1}{\nu-\mu}\int_{\mu}^{\nu} f(t)\,dt \leq \frac{f(\mu)+f(\nu)}{2}$$

where $\mu, \nu \in I$ with $\mu < \nu$. This inequality is known as Hermite-Hadamard inequality.

Various extensions and refinements of the Hermite-Hadamard inequality in the sense of classical convexity have been presented by many authors [14, 15, 41]. Last decades, the Hermite-Hadamard inequalities for abstract convexity types have been studied by many authors and attracted great interest. Log-convex, quasi-convex, $\mathbb{B}$-convex, $p$-convex functions are some of the most well-known ones[3, 6, 15, 16, 30, 47]. In recent years, the fractional operators are used to generalize the Hermite-Hadamard inequality [37]. The inequality via various fractional operators is proven for different kinds of abstract convex functions such as quasi-convex functions [4], intervalvalued coordinated convex functions [23], $\mathbb{B}^{-1}$-convex functions [24, 46], convex function with respect to a monotone function [34], exponentially nonconvex functions [40], s-convex functions [42], $\mathbb{B}$-convex functions [45] and others [10, 25].

The abstract convexity is a special sub-field of the convexity [35]. Some types of the abstract convexity have special forms and some of them are generalizations of the convexity. Nevertheless, both of them are very important and useful in application and theory, because they represent strong conditions for problems. For example, quasi-convexity which is a generalization of the convexity is applied to the optimization [13]; $\mathbb{B}$-convexity and $\mathbb{B}^{-1}$-convexity that are special forms of the convexity are applied to the economy [2, 3, 8]; $p$-convexity applied to the fixed point theory is a special form of the convexity [22]. To sum up, the convexity and abstract convexity have applications to the optimization, economy, engineering, programming, etc. [9, 31, 32, 33, 39, 43]. Inequalities based on an abstract convexity type can be applied by means of some numerical methods to the computer programming, optimization, mathematical economy, operation research, etc. [4, 5, 11, 12, 17, 26, 27, 36, 37, 42].

p-convexity, whose origin is based on p-normed spaces and sets [38, 44] is of special interest. The famous theorems of Caratheodory and Gluskin are extended for $p$-convexity in [22] and Euclidean projections of a $p$-convex body are analyzed in [19]. Fixed point theorems are studied in [18, 44]. Also, a lot more references on $p$-convexity can be given. The Hermite-Hadamard inequality for $p$-convex functions is proved in [16].

In this article, $p$-convex functions are considered in order to obtain a new version of Hermite-Hadamard inequality by applying Riemann-Liouville fractional integrals. Additionally, special cases of obtained results are presented. Finally, with an example of $p$-convex function, the results are applied and special inequalities are presented.

The paper is arranged as follows: In the second section, some introductory definitions and theorems are given in three subsections. First subsection covers the essential definitions and notations of p-convexity. Second subsection gives the definition of Riemann-Liouville fractional integrals and the last subsection recalls the Hermite-Hadamard inequalities for p-convex functions with respect to classical integral. In the third section, the inequalities of Hermite-Hadamard via fractional integrals are introduced, then their applications are given.

## 2. Preliminaries

2.1. **Definitions and Notations of $p$-convexity.** Before recalling the definitions which are the basis of the article, the following notations which will be used throughout the paper should be given.

$\mathbb{R}^n$ is the $n$-dimensional Euclidean space;

$\mathbb{R}_+^n = \{(x_1, \cdots, x_n)\,|x_j \geq 0,\ 1 \leq j \leq n\}.$

**Definition 2.2.** [7] *Let $X$ be a real vector space, $A \subset X$ and $0 < p \leq 1$. The set $A$ is called p-convex set if $\xi x+\psi y \in A$, for all $x, y \in A$ and $\xi, \psi \in [0,1]$ such that $\xi^p+\psi^p = 1$.*

Any interval of real numbers including zero or accepting zero as boundary point is a p-convex set [38].

**Definition 2.3.** [38] *Let $A \subset X$ be a p-convex set and $f : A \to \mathbb{R}$. If for all $x, y \in A$ and $\xi, \psi \in [0, 1]$ such that $\xi^p + \psi^p = 1$*

$$f(\xi x + \psi y) \leq \xi f(x) + \psi f(y), \tag{2. 1}$$

*then $f$ is called p-convex function.*

With some different notations, the inequality ( 2. 1 ) can be expressed in the following form. From the equation $\xi^p + \psi^p = 1$, if $\xi = \gamma^{\frac{1}{p}}$, $\gamma \in [0, 1]$, then the inequality ( 2. 1 ) turns into the below inequality

$$f\left(\gamma^{\frac{1}{p}}x + (1-\gamma)^{\frac{1}{p}}\,y\right) \leq \gamma^{\frac{1}{p}} f(x) + (1-\gamma)^{\frac{1}{p}}\, f(y)\ . \tag{2. 2}$$

If $\gamma = \frac{1}{2}$, then the inequality ( 2. 2 ) becomes

$$f\left(\frac{x+y}{2^{\frac{1}{p}}}\right) \leq \frac{f(x) + f(y)}{2^{\frac{1}{p}}}\ .$$

Besides, if the variables $x$ and $y$ at the above inequality are changed with $x = s\mu+(1-s)\,\nu$ and $y = s\nu + (1-s)\,\mu$ such that $s \in [0, 1]$, then it turns to the following inequality

$$f\left(\frac{\mu+\nu}{2^{\frac{1}{p}}}\right) \leq \frac{f(s\mu + (1-s)\,\nu) + f(s\nu + (1-s)\,\mu)}{2^{\frac{1}{p}}}\ . \tag{2. 3}$$

2.4. **Riemann-Liouville Fractional Integral.** In this subsection, the definition of the Riemann-Liouville fractional integral is given. Before the definitions, it is useful to remind the gamma and beta functions which are used in next section, respectively:

$$\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt \quad \text{for} \ \ \alpha > 0,$$
$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad \text{for} \ \ \alpha, \beta > 0\ .$$

**Definition 2.5.** [28, 29] *Let $f : [\mu, \nu] \to \mathbb{R}$ where $\mu, \nu \in \mathbb{R}$ such that $\mu < \nu$ and $f \in L_1[\mu, \nu]$. The left hand side of Riemann-Liouville integral $J_{\mu^+}^{\alpha} f$ and the right hand side of Riemann-Liouville integral $J_{\nu^-}^{\alpha} f$ of order $\alpha > 0$ with $\mu \geq 0$ are defined by*

$$J_{\mu^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_\mu^t (t-x)^{\alpha-1} f(x)\,dx, \quad t > \mu \tag{2. 4}$$

*and*

$$J_{\nu^-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_t^\nu (x-t)^{\alpha-1} f(x)\,dx, \quad t < \nu \tag{2. 5}$$

*respectively.*

2.6. **Hermite-Hadamard Inequality for $p$-convex Functions.** In [16], the Hermite-Hadamard Inequality for $p$-convex functions has been expressed as follows.

**Theorem 2.7.** *Let the function $f:\mathbb{R}_+\to\mathbb{R}$ be a p-convex and ir ntegrable. For $\mu,\nu\in\mathbb{R}_+$ where $\mu<\nu$, the inequality holds*

$$\begin{aligned}&2^{\frac{1}{p}-1}f\left(\frac{\mu+\nu}{2^{\frac{1}{p}}}\right)(\nu-\mu)\leq\int_\mu^\nu f(t)\,dt\\&\leq\frac{1}{2p}\left\{[\nu f(\mu)-\mu f(\nu)]\,B\left(\frac{1}{p},\frac{1}{p}\right)+p\,[\nu f(\nu)-\mu f(\mu)]\right\}\end{aligned}\tag{2. 6}$$

Besides the Hermite-Hadamard Inequality, the following generalized inequality for $p$-convex functions by substitution has proved in [16].

**Theorem 2.8.** *Let the function $f:\mathbb{R}_+\to\mathbb{R}$ be a p-convex and integrable function. For $\mu,\nu\in\mathbb{R}_+$ where $\mu<\nu$, the following inequality holds*

$$\begin{aligned}&\int_0^1 f\left((1-\gamma)^{\frac{1}{p}}\mu+\gamma^{\frac{1}{p}}\nu\right)\left(1+(1-\gamma^p)^{\frac{1}{p}-1}\gamma^{\frac{1}{p}}\right)d\gamma\\&\leq\left(1+\frac{\Gamma^2\left(\frac{1}{p}\right)}{p\Gamma\left(\frac{2}{p}\right)}\right)\left[\frac{f(\mu)+f(\nu)}{2}\right].\end{aligned}\tag{2. 7}$$

## 3. Main Results

In this section, the new Hermite-Hadamard Inequalities including the fractional integrals of Riemann-Liouville for $p$-convex functions are proved. In this process, the necessary algebraic operations require us to treat the inequality, with respect to $\alpha$ separately. Hence, we obtained the inequality for two cases: $0<\alpha\leq 1$ and $1<\alpha$. Thus, the following two separate theorems are proven depending on the cases.

**Theorem 3.1.** *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be p-convex function and $\mu,\nu\in\mathbb{R}_+$ where $\mu<\nu$ and $f\in L_1[\mu,\nu]$, provided that right and left Riemann-Liouville integrals exist. If $0<\alpha\leq 1$, then the following inequality holds*

$$\begin{aligned}&\frac{2^{\frac{1}{p}}(\nu-\mu)^\alpha}{\Gamma(\alpha+1)}f\left(\frac{\mu+\nu}{2^{\frac{1}{p}}}\right)\leq J^\alpha_{\mu^+}f(\nu)+J^\alpha_{\nu^-}f(\mu)\leq-\frac{(\nu-\mu)^{\alpha-1}\mu f(\mu)}{\Gamma(\alpha)(\alpha p-p+2)}+\\&+\frac{(\nu-\mu)^{\alpha-1}\Gamma\left(\frac{1}{p}\right)\Gamma\left(\alpha+\frac{1}{p}-1\right)}{\Gamma(\alpha)\Gamma\left(\alpha+\frac{2}{p}\right)}[(\alpha p-p+1)\nu f(\mu)-\mu f(\nu)]+\\&+\frac{(\nu-\mu)^{\alpha-1}\nu f(\nu)\Gamma(\alpha)\Gamma\left(\frac{2}{p}\right)}{p\Gamma(\alpha)\Gamma\left(\alpha+\frac{2}{p}\right)}+\frac{(-\mu)^{\alpha-1}}{2\Gamma(\alpha)}(\nu f(\nu)-\mu f(\mu))+\\&+\frac{(-\mu)^{\alpha-1}\sqrt{\pi}4^{-\frac{1}{p}}\Gamma\left(\frac{1}{p}\right)}{\Gamma(\alpha)\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}(\nu f(\mu)-\mu f(\nu)).\end{aligned}\tag{3. 8}$$

*Proof.* The first part of the inequality is obtained by multiplying the inequality ( 2. 3 ) with $s^{\alpha-1}$, $s \in (0,1]$ then integrating with respect to s over $[0,1]$ and, finally, making the substitutions $x = s\mu + (1-s)\nu$ and $x = s\nu + (1-s)\mu$ in right side of the inequality, as follows:

$$\begin{aligned}\frac{1}{\alpha} f(\frac{\mu+\nu}{2^{\frac{1}{p}}}) \leq & \frac{1}{2^{\frac{1}{p}}(\nu-\mu)^{\alpha}} \int_{\mu}^{\nu} (\nu-x)^{\alpha-1} f(x)\,dx + \\ & + \frac{1}{2^{\frac{1}{p}}(\nu-\mu)^{\alpha}} \int_{\mu}^{\nu} (x-\mu)^{\alpha-1} f(x)\,dx\end{aligned}$$

Using Definition 2.5 and properties of Gamma function, we have

$$\frac{2^{\frac{1}{p}}(\nu-\mu)^{\alpha}}{\Gamma(\alpha+1)} f\left(\frac{\mu+\nu}{2^{\frac{1}{p}}}\right) \leq J_{\mu^+}^{\alpha} f(\nu) + J_{\nu^-}^{\alpha} f(\mu) \ .$$

The second part of the inequality is proved by using some properties $0 < \gamma^{\frac{1}{p}} \leq \gamma \leq 1$ and $0 \leq (1-\gamma)^{\frac{1}{p}} \leq (1-\gamma) < 1$ for $0 < \gamma \leq 1$, $0 < p \leq 1$. The function $f(x)$ is multiplied by $(\nu-x)^{\alpha-1}$ then integrated over $x \in (\mu,\nu)$. In this stage, by setting $x = \gamma^{\frac{1}{p}}\mu + (1-\gamma)^{\frac{1}{p}}\nu$ and considering the inequality ( 2. 2 ), we can write

$$\begin{aligned}& \int_0^1 \left(\nu - \gamma^{\frac{1}{p}}\mu - (1-\gamma)^{\frac{1}{p}}\nu\right)^{\alpha-1} f(\gamma^{\frac{1}{p}}\mu + (1-\gamma)^{\frac{1}{p}}\nu) \\ & \quad \frac{1}{p}\left(-\gamma^{\frac{1}{p}-1}\mu + (1-\gamma)^{\frac{1}{p}-1}\nu\right) d\gamma \\ & \leq \int_0^1 \left(\nu - \gamma^{\frac{1}{p}}\mu - (1-\gamma)^{\frac{1}{p}}\nu\right)^{\alpha-1} \left(\gamma^{\frac{1}{p}} f(\mu) + (1-\gamma)^{\frac{1}{p}} f(\nu)\right) \\ & \quad \frac{1}{p}\left(-\gamma^{\frac{1}{p}-1}\mu + (1-\gamma)^{\frac{1}{p}-1}\nu\right) d\gamma \\ & \leq \int_0^1 \gamma^{\alpha-1}(\nu-\mu)^{\alpha-1} \left(\gamma^{\frac{1}{p}} f(\mu) + (1-\gamma)^{\frac{1}{p}} f(\nu)\right) \\ & \quad \frac{1}{p}\left(-\gamma^{\frac{1}{p}-1}\mu + (1-\gamma)^{\frac{1}{p}-1}\nu\right) d\gamma \\ & = -\frac{1}{p}(\nu-\mu)^{\alpha-1}\mu f(\mu) \int_0^1 \gamma^{\alpha+\frac{2}{p}-2}\gamma d\gamma \\ & \quad + \frac{1}{p}(\nu-\mu)^{\alpha-1}\nu f(\mu) \int_0^1 \gamma^{\alpha+\frac{1}{p}-1}(1-\gamma)^{\frac{1}{p}-1} d\gamma \\ & \quad - \frac{1}{p}(\nu-\mu)^{\alpha-1}\mu f(\nu) \int_0^1 \gamma^{\alpha+\frac{1}{p}-2}(1-\gamma)^{\frac{1}{p}} d\gamma\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{p}(\nu-\mu)^{\alpha-1}\nu f(\nu)\int_0^1 \gamma^{\alpha-1}(1-\gamma)^{\frac{2}{p}-1}d\gamma \\
&=-\frac{(\nu-\mu)^{\alpha-1}\mu f(\mu)}{\alpha p-p+2}+\frac{(\nu-\mu)^{\alpha-1}\nu f(\nu)\Gamma(\alpha)\Gamma\left(\frac{2}{p}\right)}{p\Gamma\left(\alpha+\frac{2}{p}\right)} \\
&+\frac{(\nu-\mu)^{\alpha-1}\Gamma\left(\frac{1}{p}\right)\Gamma\left(\alpha+\frac{1}{p}-1\right)}{\Gamma\left(\alpha+\frac{2}{p}\right)}\left[(\alpha p-p+1)\nu f(\mu)-\mu f(\nu)\right].
\end{aligned}
$$

Now, if the function $f(x)$ is multiplied by $(x-\mu)^{\alpha-1}$ and integrated over $x \in (\mu,\nu)$ using the same substitution and the inequality ( 2. 2 ), then

$$
\begin{aligned}
&\int_0^1 \left(\gamma^{\frac{1}{p}}\mu+(1-\gamma)^{\frac{1}{p}}\nu-\mu\right)^{\alpha-1} f(\gamma^{\frac{1}{p}}\mu+(1-\gamma)^{\frac{1}{p}}\nu) \\
&\quad \frac{1}{p}\left(-\gamma^{\frac{1}{p}-1}\mu+(1-\gamma)^{\frac{1}{p}-1}\nu\right)d\gamma \\
&\le \int_0^1 \left(\gamma^{\frac{1}{p}}\mu+(1-\gamma)^{\frac{1}{p}}\nu-\mu\right)^{\alpha-1}\left(\gamma^{\frac{1}{p}}f(\mu)+(1-\gamma)^{\frac{1}{p}}f(\nu)\right) \\
&\quad \frac{1}{p}\left(-\gamma^{\frac{1}{p}-1}\mu+(1-\gamma)^{\frac{1}{p}-1}\nu\right)d\gamma \\
&\le \int_0^1 (-\mu)^{\alpha-1}\left(\gamma^{\frac{1}{p}}f(\mu)+(1-\gamma)^{\frac{1}{p}}f(\nu)\right) \\
&\quad \frac{1}{p}\left(-\gamma^{\frac{1}{p}-1}\mu+(1-\gamma)^{\frac{1}{p}-1}\nu\right)d\gamma \\
&=-\frac{1}{p}(-\mu)^{\alpha-1}\mu f(\mu)\int_0^1 \gamma^{\frac{2}{p}-1}d\gamma+\frac{1}{p}(-\mu)^{\alpha-1}\nu f(\mu)\int_0^1 \gamma^{\frac{1}{p}}(1-\gamma)^{\frac{1}{p}-1}d\gamma \\
&\quad -\frac{1}{p}(-\mu)^{\alpha-1}\mu f(\nu)\int_0^1 \gamma^{\frac{1}{p}-1}(1-\gamma)^{\frac{1}{p}}d\gamma \\
&\quad +\frac{1}{p}(-\mu)^{\alpha-1}\nu f(\nu)\int_0^1 (1-\gamma)^{\frac{2}{p}-1}d\gamma \\
&=\frac{(-\mu)^{\alpha-1}}{2}(\nu f(\nu)-\mu f(\mu))+\frac{(-\mu)^{\alpha-1}\sqrt{\pi}4^{-\frac{1}{p}}\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}(\nu f(\mu)-\mu f(\nu)) \ .
\end{aligned}
$$

Finally, if the two inequalities above are summed and the both sides of the inequality are multiplied by $\frac{1}{\Gamma(\alpha)}$, then desired result of the right side of ( 3. 8 ) is obtained. □

Now, the case of $\alpha \ge 1$ is proved in the next theorem. While this case affects the right side of the inequality ( 3. 8 ), the left side remains same.

**Theorem 3.2.** *Let $f:\mathbb{R}_+ \to \mathbb{R}_+$ be p-convex function and $\mu,\nu \in \mathbb{R}_+$ where $\mu<\nu$ and $f \in L_1[\mu,\nu]$, provided that right and left Riemann-Liouville integrals exist. If $1<\alpha$, then* *the following inequality holds:*

$$\frac{2^{\frac{1}{p}}(\nu-\mu)^{\alpha}}{\Gamma(\alpha+1)} f\left(\frac{\mu+\nu}{2^{\frac{1}{p}}}\right) \leq J_{\mu^{+}}^{\alpha} f(\nu)+J_{\nu^{-}}^{\alpha} f(\mu) \leq \frac{\nu^{\alpha-1}(\nu f(\nu)-\mu f(\mu))}{2\Gamma(\alpha)}+$$
$$+\frac{\nu^{\alpha-1}\sqrt{\pi}4^{-\frac{1}{p}}\Gamma\left(\frac{1}{p}\right)(\nu f(\mu)-\mu f(\nu))}{p\Gamma(\alpha)\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}-\frac{(\nu-\mu)^{\alpha-1}\mu f(\mu)\Gamma\left(\frac{2}{p}\right)}{p\Gamma\left(\alpha+\frac{2}{p}\right)}+$$
$$+\frac{(\nu-\mu)^{\alpha-1}\Gamma\left(\frac{1}{p}\right)\Gamma\left(\alpha+\frac{1}{p}-1\right)(\nu f(\mu)-\mu f(\nu)(\alpha p-p+1))}{\Gamma\left(\alpha+\frac{2}{p}\right)\Gamma(\alpha)}+\frac{(\nu-\mu)^{\alpha-1}\nu f(\nu)}{((\alpha-1)p+2)\Gamma(\alpha)}\,. \tag{3. 9}$$

*Proof.* Using the same method in the proof of Theorem 3.1, we have the left-hand side of ( 3. 9 ). Let us prove the other side. The inequalities

$$\begin{aligned}&\int_0^1\left(\nu-\gamma^{\frac{1}{p}}\mu-(1-\gamma)^{\frac{1}{p}}\nu\right)^{\alpha-1} f(\gamma^{\frac{1}{p}}\mu+(1-\gamma)^{\frac{1}{p}}\nu)\\ &\quad\frac{1}{p}\left(-\gamma^{\frac{1}{p}-1}\mu+(1-\gamma)^{\frac{1}{p}-1}\nu\right)d\gamma\\ &\leq\int_0^1\left(\nu-\gamma^{\frac{1}{p}}\mu-(1-\gamma)^{\frac{1}{p}}\nu\right)^{\alpha-1}\left(\gamma^{\frac{1}{p}}f(\mu)+(1-\gamma)^{\frac{1}{p}}f(\nu)\right)\\ &\quad\frac{1}{p}\left(-\gamma^{\frac{1}{p}-1}\mu+(1-\gamma)^{\frac{1}{p}-1}\nu\right)d\gamma\end{aligned}$$

and

$$\begin{aligned}&\int_0^1\left(\gamma^{\frac{1}{p}}\mu+(1-\gamma)^{\frac{1}{p}}\nu-\mu\right)^{\alpha-1} f(\gamma^{\frac{1}{p}}\mu+(1-\gamma)^{\frac{1}{p}}\nu)\\ &\quad\frac{1}{p}\left(-\gamma^{\frac{1}{p}-1}\mu+(1-\gamma)^{\frac{1}{p}-1}\nu\right)d\gamma\\ &\leq\int_0^1\left(\gamma^{\frac{1}{p}}\mu+(1-\gamma)^{\frac{1}{p}}\nu-\mu\right)^{\alpha-1}\left(\gamma^{\frac{1}{p}}f(\mu)+(1-\gamma)^{\frac{1}{p}}f(\nu)\right)\\ &\quad\frac{1}{p}\left(-\gamma^{\frac{1}{p}-1}\mu+(1-\gamma)^{\frac{1}{p}-1}\nu\right)d\gamma\end{aligned}$$

come from the first proof. If the inequalities are summed and multiplied by $\frac{1}{\Gamma(\alpha)}$, then

$$\begin{aligned}&J_{\mu^{+}}^{\alpha} f(\nu)+J_{\nu^{-}}^{\alpha} f(\mu)\\ &\leq\frac{1}{p\Gamma(\alpha)}\int_0^1\left[\left(\nu-\gamma^{\frac{1}{p}}\mu-(1-\gamma)^{\frac{1}{p}}\nu\right)^{\alpha-1}+\left(\gamma^{\frac{1}{p}}\mu+(1-\gamma)^{\frac{1}{p}}\nu-\mu\right)^{\alpha-1}\right]\\ &\quad\left(-\mu f(\mu)\gamma^{\left(\frac{2}{p}-1\right)}+\nu f(\mu)\gamma^{\frac{1}{p}}(1-\gamma)^{\left(\frac{1}{p}-1\right)}-\right.\\ &\quad\left.-\mu f(\nu)\gamma^{\left(\frac{1}{p}-1\right)}(1-\gamma)^{\frac{1}{p}}+\nu f(\nu)(1-\gamma)^{\left(\frac{2}{p}-1\right)}\right)d\gamma\end{aligned} \tag{3. 10}$$

is obtained. Since $1<\alpha<+\infty$, $0\leq\gamma^{\frac{1}{p}}\leq\gamma\leq 1$ and $0\leq(1-\gamma)^{\frac{1}{p}}\leq(1-\gamma)\leq 1$, one has $\left(\nu-\gamma^{\frac{1}{p}}\mu-(1-\gamma)^{\frac{1}{p}}\nu\right)^{\alpha-1}\leq\nu^{\alpha-1}$ and
$\left(\gamma^{\frac{1}{p}}\mu+(1-\gamma)^{\frac{1}{p}}\nu-\mu\right)^{\alpha-1}\leq(\nu-\mu)^{\alpha-1}(1-\gamma)^{\alpha-1}$. Hence, the inequality ( 3. 10 ) is handled as

$$
\begin{aligned}
&J_{\mu^+}^{\alpha} f(\nu) + J_{\nu^-}^{\alpha} f(\mu) \\
&\leq \frac{1}{p\Gamma(\alpha)} \int_0^1 \left[ \left( \nu - \gamma^{\frac{1}{p}} \mu - (1-\gamma)^{\frac{1}{p}} \nu \right)^{\alpha-1} + \left( \gamma^{\frac{1}{p}} \mu + (1-\gamma)^{\frac{1}{p}} \nu - \mu \right)^{\alpha-1} \right] \\
&\Big( -\mu f(\mu) \gamma^{\left(\frac{2}{p}-1\right)} + \nu f(\mu) \gamma^{\frac{1}{p}} (1-\gamma)^{\left(\frac{1}{p}-1\right)} \\
&-\mu f(\nu) \gamma^{\left(\frac{1}{p}-1\right)} (1-\gamma)^{\frac{1}{p}} + \nu f(\nu) (1-\gamma)^{\left(\frac{2}{p}-1\right)} \Big) d\gamma \\
&\leq \frac{1}{p\Gamma(\alpha)} \int_0^1 \left[ \nu^{\alpha-1} + (1-\gamma)^{\alpha-1} (\nu-\mu)^{\alpha-1} \right] \\
&\Big( -\mu f(\mu) \gamma^{\left(\frac{2}{p}-1\right)} + \nu f(\mu) \gamma^{\frac{1}{p}} (1-\gamma)^{\left(\frac{1}{p}-1\right)} \\
&-\mu f(\nu) \gamma^{\left(\frac{1}{p}-1\right)} (1-\gamma)^{\frac{1}{p}} + \nu f(\nu) (1-\gamma)^{\left(\frac{2}{p}-1\right)} \Big) d\gamma \\
&= -\frac{1}{p\Gamma(\alpha)} \nu^{\alpha-1} \mu f(\mu) \int_0^1 \gamma^{\left(\frac{2}{p}-1\right)} d\gamma + \frac{1}{p\Gamma(\alpha)} \nu^{\alpha} f(\mu) \int_0^1 \gamma^{\frac{1}{p}} (1-\gamma)^{\left(\frac{1}{p}-1\right)} d\gamma \\
&\quad - \frac{1}{p\Gamma(\alpha)} \nu^{\alpha-1} \mu f(\nu) \int_0^1 \gamma^{\left(\frac{1}{p}-1\right)} (1-\gamma)^{\frac{1}{p}} d\gamma \\
&\quad + \frac{1}{p\Gamma(\alpha)} \nu^{\alpha} f(\nu) \int_0^1 (1-\gamma)^{\left(\frac{2}{p}-1\right)} d\gamma \\
&\quad - \frac{1}{p\Gamma(\alpha)} (\nu-\mu)^{\alpha-1} \mu f(\mu) \int_0^1 (1-\gamma)^{\alpha-1} \gamma^{\left(\frac{2}{p}-1\right)} d\gamma \\
&\quad + \frac{1}{p\Gamma(\alpha)} (\nu-\mu)^{\alpha-1} \nu f(\mu) \int_0^1 \gamma^{\frac{1}{p}} (1-\gamma)^{\left(\alpha+\frac{1}{p}-2\right)} d\gamma \\
&\quad - \frac{1}{p\Gamma(\alpha)} (\nu-\mu)^{\alpha-1} \mu f(\nu) \int_0^1 \gamma^{\left(\frac{1}{p}-1\right)} (1-\gamma)^{\alpha+\frac{1}{p}-1} d\gamma \\
&\quad + \frac{1}{p\Gamma(\alpha)} (\nu-\mu)^{\alpha-1} \nu f(\nu) \int_0^1 (1-\gamma)^{\left(\alpha+\frac{2}{p}-2\right)} d\gamma \\
&= -\frac{1}{p\Gamma(\alpha)} \nu^{\alpha-1} \mu f(\mu) \frac{p}{2} + \frac{1}{p\Gamma(\alpha)} \nu^{\alpha} f(\mu) \frac{\sqrt{\pi} 4^{-\frac{1}{p}} \Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)} \\
&\quad - \frac{1}{p\Gamma(\alpha)} \nu^{\alpha-1} \mu f(\nu) \frac{\sqrt{\pi} 4^{-\frac{1}{p}} \Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)} \\
&\quad + \frac{1}{p\Gamma(\alpha)} \nu^{\alpha} f(\nu) \frac{p}{2} - \frac{1}{p\Gamma(\alpha)} (\nu-\mu)^{\alpha-1} \mu f(\mu) \frac{\Gamma(\alpha) \Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\alpha+\frac{2}{p}\right)} \\
&\quad + \frac{1}{p\Gamma(\alpha)} (\nu-\mu)^{\alpha-1} \nu f(\mu) \frac{\Gamma\left(1+\frac{1}{p}\right) \Gamma\left(\alpha+\frac{1}{p}-1\right)}{\Gamma\left(\alpha+\frac{2}{p}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{p\Gamma(\alpha)}(\nu-\mu)^{\alpha-1}\mu f(\nu)\frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma\left(\alpha+\frac{2}{p}\right)}\\
&+\frac{1}{p\Gamma(\alpha)}(\nu-\mu)^{\alpha-1}\nu f(\nu)\frac{p}{(\alpha-1)p+2}\\
=&\frac{\nu^{\alpha-1}(\nu f(\nu)-\mu f(\mu))}{2\Gamma(\alpha)}+\frac{\nu^{\alpha-1}\sqrt{\pi}4^{-\frac{1}{p}}\Gamma\left(\frac{1}{p}\right)(\nu f(\mu)-\mu f(\nu))}{p\Gamma(\alpha)\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}\\
&-\frac{(\nu-\mu)^{\alpha-1}\mu f(\mu)\Gamma\left(\frac{2}{p}\right)}{p\Gamma\left(\alpha+\frac{2}{p}\right)}+\frac{(\nu-\mu)^{\alpha-1}\nu f(\nu)}{((\alpha-1)p+2)\Gamma(\alpha)}\\
&+\frac{(\nu-\mu)^{\alpha-1}\Gamma\left(\frac{1}{p}\right)\Gamma\left(\alpha+\frac{1}{p}-1\right)(\nu f(\mu)-\mu f(\nu)(\alpha p-p+1))}{\Gamma\left(\alpha+\frac{2}{p}\right)\Gamma(\alpha)}.
\end{aligned}
$$

This proves the theorem. □

**Corollary 3.3.** *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be $p$-convex function and $\mu,\nu\in\mathbb{R}_+$ where $\mu<\nu$ and $f\in L_1[\mu,\nu]$. The following inequality holds:*

$$
\begin{aligned}
2^{\frac{1}{p}-1}f\left(\frac{\mu+\nu}{2^{\frac{1}{p}}}\right)(\nu-\mu)\leq\int_\mu^\nu f(x)\,dx\leq&\frac{\nu f(\nu)-\mu f(\mu)}{2}\\
&+\frac{3}{4}B\left(\frac{1}{p},\frac{1}{p}\right)(\nu f(\mu)-\mu f(\nu))\ . \qquad (3.\ 11)
\end{aligned}
$$

*Proof.* This can be proved by taking $\alpha=1$ in Theorem 3.1. □

**Corollary 3.4.** *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be $p$-convex function and $\mu,\nu\in\mathbb{R}_+$ where $\mu<\nu$ and $f\in L_1[\mu,\nu]$. The following inequality holds:*

$$
\begin{aligned}
2^{\frac{1}{p}-1}f\left(\frac{\mu+\nu}{2^{\frac{1}{p}}}\right)(\nu-\mu)\leq\int_\mu^\nu f(x)\,dx\leq&\frac{\nu f(\nu)-\mu f(\mu)}{2}\\
&+\frac{p^2+1}{4p}B\left(\frac{1}{p},\frac{1}{p}\right)(\nu f(\mu)-\mu f(\nu))\ . \qquad (3.\ 12)
\end{aligned}
$$

*Proof.* The inequality ( 3. 12 ) is obtained by taking $\alpha=1$ in the inequality ( 3. 8 ). □

Consequently, some new type inequalities are proved, which are different from the inequalities given in [16].

Here, an example of $p$-convex function is given. Then, this function is applied to the obtained inequalities.

**Example 3.5.** *Let $f:\mathbb{R}_+\to\mathbb{R}$ be the function $f(x)=x^2$. The function $f$ is $p$-convex. Indeed, for $0<p\leq 1$*

$$
\begin{aligned}
f(\delta^{\frac{1}{p}}x+(1-\delta)^{\frac{1}{p}}y) &= \left(\delta^{\frac{1}{p}}x+(1-\delta)^{\frac{1}{p}}y\right)^2 \\
&= \delta^{\frac{2}{p}}x^2+2\delta^{\frac{1}{p}}(1-\delta)^{\frac{1}{p}}xy+(1-\delta)^{\frac{2}{p}}y^2 \\
&\leq \delta^{\frac{2}{p}}x^2+\delta^{\frac{1}{p}}(1-\delta)^{\frac{1}{p}}\left(x^2+y^2\right)+(1-\delta)^{\frac{2}{p}}y^2 \\
&= \delta^{\frac{1}{p}}x^2\left(\delta^{\frac{1}{p}}+(1-\delta)^{\frac{1}{p}}\right)+(1-\delta)^{\frac{1}{p}}y^2\left(\delta^{\frac{1}{p}}+(1-\delta)^{\frac{1}{p}}\right) \\
&\leq \delta^{\frac{1}{p}}x^2+(1-\delta)^{\frac{1}{p}}y^2=\delta^{\frac{1}{p}}f(x)+(1-\delta)^{\frac{1}{p}}f(y)\ .
\end{aligned}
$$

For the function $f:\mathbb{R}_+\to\mathbb{R}$, $f(x)=x^2$, Corollary 3.3 gives the following inequality:

$$
2^{\frac{1}{p}-1}\left(\frac{\mu+\nu}{2^{\frac{1}{p}}}\right)^2(\nu-\mu)\leq\int_\mu^\nu x^2dx\leq\frac{\nu^3-\mu^3}{2}+\frac{3}{4}B\left(\frac{1}{p},\frac{1}{p}\right)\left(\nu\mu^2-\mu\nu^2\right)
$$

$$
\frac{(\mu+\nu)^2}{2^{\frac{1}{p}+1}}\leq\frac{\nu^2+\mu\nu+\mu^2}{3}\leq\frac{\nu^2+\mu\nu+\mu^2}{2}-\frac{3}{4}B\left(\frac{1}{p},\frac{1}{p}\right)\mu\nu\ . \tag{3. 13}
$$

In Corollary 3.4, the inequality below is obtained by getting the function $f(x)=x^2$.

$$
\frac{(\mu+\nu)^2(\nu-\mu)}{2^{\frac{1}{p}+1}}\leq\frac{\nu^3-\mu^3}{3}\leq\frac{\nu^3-\mu^3}{2}+\frac{p^2+1}{4p}B\left(\frac{1}{p},\frac{1}{p}\right)\mu\nu(\mu-\nu)\,. \tag{3. 14}
$$

Also, in this application, it can be easily seen the verification of the inequality ( 3. 14 ) by calculating for the same specific $\mu$ and $\nu$ as above.

## 4. CONCLUSION

In this paper, a new abstract convexity type p-convexity is handled. This convexity type was studied by J. Bastero, J. Bernues, A. Pena for sets in 1995. There were some other papers about $p$-convex sets. The $p$-convex functions were introduced by Sezer, and others in 2021. They also gave the inequality of Hermite-Hadamard for $p$-convex functions in 2021.

In this study, the generalized Hermite-Hadamard inequality for $p$-convex functions including the fractional integral of Riemann-Liouville is proved. To give these inequalities, the degree of the fractional integral should be examined separately as $0<\alpha\leq 1$ and $1\leq\alpha<+\infty$. Additionally, corollaries are obtained about these theorems. Finally, some applications according to the corollaries are given.

## REFERENCES

[1] S. Abramovic, G. Jameson, G. Sinnamon, *Refining Jensen's Inequality*, Bulletin Mathematique de la Societe des Sciences and Mathematiques de Roumanie (2004), 3-14.

[2] G. Adilov, I. Yesilce, $\mathbb{B}^{-1}-$*convex Functions*, Journal of Convex Analysis **24**, No. 2 (2017) 505-517.

[3] G. Adilov, I. Yesilce, *On Generalization of the Concept of Convexity*, Hacettepe Journal of Mathematics and Statistics **41**, No. 5 (2012) 723-730.

[4] N. Alp, M. Z. Sarkaya, M. Kunt, I. Iscan, *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, Journal of King Saud University-Science **30**, No. 2 (2018) 193-203.

[5] Y. Atalan, *On a New Fixed Point Iterative Algorithm For General Variational Inequalities*, Journal of Nonlinear and Convex Analysis **20**, No. 11 (2019) 2371-2386.

[6] M.U. Awan, N. Akhtar, S. Iftikhar, M. A. Noor, Y. Chu, *New HermiteHadamard type inequalities for n-polynomial harmonically convex functions*. Journal of Inequalities and Applications **2020**, (2020) 125.

[7] J. Bastero, J. Bernues, A. Pena, *The Theorems of Caratheodory and Gluskin for* $0 < p < 1$. Proceedings of the American Mathematical Society (1995) 141-144.

[8] W. Briec, C. D. Horvath, $\mathbb{B}$*-convexity*, Optimization **53**, (2004) 107-127.

[9] W. Briec, Q. B. Liang, *On some semilattice structures for production technologies*, Eur. J. Oper. Res. **215**, (2011) 740749.

[10] H. Budak, H. Kara, M. Ali, S. Khan, Y. Chu, *Fractional Hermite-Hadamard-type inequalities for interval-valued co-ordinated convex functions*. Open Mathematics **19**, No.1 (2021) 1081-1097.

[11] S. I. Butt, M. Umar, S. Rashid, A. O. Akdemir, Y. Chu *New HermiteJensenMercer-type inequalities via k-fractional integrals*, Advances in Difference Equations **2020** (2020), 635.

[12] H. Chu, S. Rashid, Z. Hammouch, Y. Chu, *New fractional estimates for Hermite-Hadamard-Mercers type inequalities*, Alexandria Engineering Journal **59**, No. 5 (2020) 3079-3089.

[13] J. A. dos Santos Gromicho, *Quasiconvex Optimization and Location Theory*, **9** Springer Science and Business Media, 2013.

[14] S. S. Dragomir, *An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math **3**, No. 2 (2002) 31.

[15] S. S. Dragomir, C. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, Science Direct Working Paper, (S1574-0358), 04, 2003.

[16] Z. Eken, S. Kemali, G. Tinaztepe, G. Adilov, *The Hermite-Hadamard Inequalities for p-Convex Functions*, Hacettepe Journal of Mathematics and Statistics **50**, No. 5 (2021) 1268-1279.

[17] K. J. Forster, K. Petras, *On a Problem Proposed by H. Brass Concerning the Remainder Term in Quadrature for Convex Functions*, Numerische Mathematik **57**, No. 1 (1990) 763-777.

[18] L. Gholizadeh, E. Karapnar, M. Roohi, *Some fixed point theorems in locally p-convex spaces*, Fixed Point Theory and Applications **2013**, No. 1 (2013) 312.

[19] O. Guedon, A. E. Litvak, *Euclidean projections of a* $p$*-convex body*, In Geometric aspects of functional analysis, (2000), 95-108. Springer, Berlin, Heidelberg.

[20] J. Hadamard, *Etude sur les proprietes des fonctions entieres et en particulier d'une fonction consideree par Riemann*, Journal des Mathematiques Pures et Appliquees **58** (1893), 171-215.

[21] C. Hermite, *Sur deux limites d'une integrale define* Mathesis **3** (1883), 82.

[22] J. B. Jesus Bastero, A. Pena, *The Theorems of Caratheodory and Gluskin for* $0 < p < 1$, Proceedings of the American Mathematical Society **123**, No. 1 (1995) 141-144.

[23] H. Kara, M. A. Ali, H. Budak, *HermiteHadamardtype inequalities for intervalvalued coordinated convex functions involving generalized fractional integrals*, Math Meth Appl Sci. **44** (2021) 104 123.

[24] S. Kemali, G. Tinaztepe, G. Adilov, *New Type Inequalities for* $\mathbb{B}^{-1}$*-convex Functions involving Hadamard Fractional Integral*, Facta Univ.-series: Math. Informa **33**, (2018) 697-704.

[25] M. A. Khan, Y. Khurshid, T.-S. Du, Y. Chu, *Generalization of Hermite-Hadamard Type Inequalities via Conformable Fractional Integrals*, Journal of Function Spaces **2018** (2018) Article ID 5357463, 12 pages.

[26] Y. Khurshid, M. A. Khan, Y. Chu, *Conformable integral version of Hermite-Hadamard-Fejr inequalities via* $\eta$*-convex functions*, AIMS Mathematics **5**, No.5 (2020) 5106-5120.

[27] Y. Khurshid, M. A. Khan, Y. Chu, Z. A. Khan, *Hermite-Hadamard-Fejr Inequalities for Conformable Fractional Integrals via Preinvex Functions*, Journal of Function Spaces **2019**, (2019) Article ID 3146210, 9 pages.

[28] A. A. Kilbas, O. I. Marichev, S. G. Samko, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Switzerland 1993.

[29] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Vol. 204. Elsevier 2006.

[30] M.A. Latif, S. Rashid, S.S. Dragomir, Y. Chu, *HermiteHadamard type inequalities for co-ordinated convex and qausi-convex functions and their applications*, Journal of Inequalities and Applications **2019** (2019) 317.

[31] S. Maldar, *Iterative algorithms of generalized nonexpansive mappings and monotone operators with application to convex minimization problem*, Journal of Applied Mathematics and Computing **68**, No. 3 (2022) 1841-1868.

[32] S. Maldar, F. Grsoy, Y. Atalan, M. Abbas, *On a three-step iteration process for multivalued Reich-Suzuki type $\alpha$ nonexpansive and contractive mappings*, Journal of Applied Mathematics and Computing **68**, No. 2 (2022) 863-883.

[33] S. Maldar, V. Karakaya, *Convergence of Jungck-Kirk type iteration method with applications*, Punjab University Journal of Mathematics **53**, No. 2 (2022) 75-87.

[34] P. Mohammed, *HermiteHadamard inequalities for RiemannLiouville fractional integrals of a convex function with respect to a monotone function*, Math Meth Appl Sci. **44**, (2021) 23142324.

[35] A. Rubinov, *Abstract Convexity and Global Optimization*, Boston-Dordrecht-London, Kluwer Academic Publishers, 2000.

[36] M. Z. Sarikaya, H. Budak, *New inequalities of Opial type for conformable fractional integrals*, Turkish Journal of Mathematics **41**, No. 5 (2017) 1164-1173.

[37] E. Set, M. Z. Sarikaya, M. E. Ozdemir, H. Yldrm, *The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results*, Journal of Applied Mathematics, Statistics and Informatics **10**, No.2 (2014) 69-83.

[38] S. Sezer, Z. Eken, G. Tinaztepe, G. Adilov, *p-convex Functions and Their Some Properties*, Numerical Functional Analysis and Optimization **42**, No. 4 (2021) 443-459.

[39] I. Singer, *Abstract Convex Analysis*, New York, John Wiley and Sons, 1997.

[40] H. M. Srivastava, A. Kashuri, P. O. Mohammed, D. Baleanu, Y. S. Hamed, *Fractional integral inequalities for exponentially nonconvex functions and their applications*, Fractal and Fractional **5**, No. 3 (2021) 80.

[41] H. M. Srivastava, Z. H. Zhang, Y. D. Wu, *Some further refinements and extensions of the HermiteHadamard and Jensen inequalities in several variables*, Mathematical and computer modelling **54**, No. 11-12 (2011) 2709-2717.

[42] F. Usta, H. Budak, M. Z. Sarikaya, E. Set, *On generalization of trapezoid type inequalities for s-convex functions with generalized fractional integral operators*, Filomat **32**, No. 6 (2018) 2153-2171.

[43] M. L. J. Van De Vel, *Theory of Convex Structures*, Vol. 50. North-Holland, Amsterdam, North Holland Mathematical Library, 1993.

[44] J. Z. Xiao, X. H. Zhu, *Some fixed point theorems for s-convex subsets in p-normed spaces*, Nonlinear Analysis: Theory, Methods and Applications **74**, No. 5 (2011) 1738-1748.

[45] I. Yesilce, G. Adilov, *Fractional integral inequalities for $\mathbb{B}$-convex functions*, Creative Mathematics and Informatics **26**, No. 3 (2017) 345-351.

[46] I. Yesilce, *Some Inequalities for $\mathbb{B}^{-1}$-convex Functions via Fractional Integral Operator*, TWMS Journal of Applied and Engineering Mathematics **9**, No. 3 (2019) 620-625.

[47] S. Zhou, S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, Y. Chu, *New Hermite-Hadamard type inequalities for exponentially convex functions and applications*, AIMS Mathematics **5**, No.6 (2020) 6874-6901.