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On Banach Common Fixed Point Results in Symmetric G_d-Metric Spaces

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Abstract. In this paper, common fixed-point problems for locally contractive mappings in symmetric G_d-metric like spaces are proved and error bounds are disscussed in detail. Besides, we gave examples to validate main results. In our research, we have proved that a pair of self-mappings satisfies Banach contraction, in local domain instead of global domain. Also, application of G_d-metric like spaces for solving Urysohn integral equations is given.

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1. Introduction

Self-mappings F_1 and F_2 on S are Banach contraction mappings if $d(F_1t, F_2u) \leq \xi d(t, u)$ satisfies for all $t, u \in S$ where $0 \leq \xi < 1$. As of late, numerous outcomes showed up in writing related to fixed point brings about complete metric spaces supplied with a halfway requesting. Ran and Reurings [13] demonstrated a simple case of Banach's fixed point hypothesis in partial ordered metric space and offered applications to matrix equations. Subsequently, Nieto et. al. [11] expanded the outcomes for non-decreasing functions and applied this outcome to acquire the solution for the first order ordinary differential equation with periodic boundary conditions. In [10], Z. Mustafa and B. Sims introduced an alternative more robust generalization of metric spaces called G-metric spaces. They did this to overcome fundamental flaws in B. C. Dhage's [7] theory of generalized metric spaces, flaws that invalidate most of the results claimed for these spaces. During the sixties, 2-metric spaces were introduced by Gahler. A 2-metric is a generalization of the usual notion of a metric, but different authors proved that there is no relation between these two functions. For instance, Ha et al. in show that a 2-metric need not be a continuous function of its variables, whereas an ordinary metric is. Further, there is no easy relationship between results obtained in the two settings, in particular the contraction mapping theorem in metric spaces and in 2-metric spaces are unrelated. These considerations led Bapure Dhage in his PhD thesis to introduce a new class of generalized metrics called D-metrics. Recently, Arshad [4] demonstrated fixed points of a mapping fulfilling a contractive condition on closed ball (C_B) in ordered complete partial metric space. The overwhelmed mapping [4] which fulfills the condition $\digamma_1 u \preceq u$ happens normally. For example, u signifies the aggregate amount of wheat cultivated over a specific period and $\digamma_1(u)$ gives the amount of wheat used over a similar period in a specific town, at that point we should have $\digamma_1 u \preceq u$. To access more comprehensive information, readers may refer to the following articles [3, 5, 6, 7, 8, 12, 14, 15, 16] and the associated references provided in them.

This article presents the demonstration of fixed point existence for two self-mappings that operate within a closed ball (C_B), a subset of S, and conform to a Banach type contractive condition. The primary focus of the article is to establish the presence of fixed points within C_B. A significant feature of the article is the adoption of local contractive conditions as opposed to global contractive conditions. This particular approach is employed to derive fixed point results that are both unique and common. To verify the reliability and soundness of the results, the article includes a range of examples that serve to validate the obtained outcomes.

2. Preliminaries

Here are some essential preliminary points to enhance readers' convenience and understanding before delving into the topic.

Definition 2.1. [4] *In partial ordered set $(S, \preceq)$, elements $u, v \in S$ are called comparable elements if either $u \preceq v$ or $v \preceq u$.*

Definition 2.2. [4] *In partial ordered set $(S, \preceq)$, a self-mapping $\digamma_1$ on S, is said to be dominated if $\digamma_1 u \preceq u$ for each $u \in S$.*

Definition 2.3. [9] *In partial ordered set $(S, \preceq)$, a self-mapping $\digamma_1$ on S, is said to be dominating if $u \preceq \digamma_1 u$ for each $u \in S$.*

Shoaib et al. [10] employed the principles of generalized metric spaces and dislocated generalized metric spaces to establish a range of fixed point theorems. The authors begin by providing clear definitions of these fundamental concepts, namely the generalized metric space and the dislocated generalized metric space. These definitions serve to enhance the understanding of these key concepts and lay the foundation for the subsequent theoretical developments presented in the paper.

Definition 2.4. [10] *Let $S \neq \phi$, and mapping $G : S \times S \times S \longrightarrow [0, \infty)$ satisfying;*

(G1) $G(t, u, v) = 0$ *if* $v = u = t$,

(G2) $G(t, t, u) > 0$, *for all* $t, u \in S$ *with* $t \neq u$,

(G3) $G(u,u,v) \leq G(t,u,v)$, *for all* $u,v,t \in S$ *and* $t \neq v$,

(G4) $G(t,u,v) = G(u,t,v) = G(u,v,t) = G(t,v,u) = G(v,t,u) = G(v,u,t)$, *(symmetry in all three variables),*

(G5) $G(u,v,t) = G(u,u,w) + G(w,v,t)$, *for all* $w,t,u,v \in S$, *(rectangular inequality), is called a G-metric* (GM) *and the pair* (S,G) *is called a G-metric space* (G_{MS})*. The function G is continuous in three of its variables.*

Definition 2.5. [10] *Let* $S \neq \phi$, *and mapping* $G_d : S \times S \times S \longrightarrow [0,\infty)$ *satisfying;*

(i) $G_d(u,v,t) = G_d(u,t,v) = G_d(t,u,v) = G_d(t,v,u) = G_d(v,t,u) = G_d(v,u,t) = 0$, *then* $u = v = t$,

(ii) $G_d(u,v,t) \leq G_d(u,u,w) + G_d(w,v,t)$, *for all* $w,t,u,v \in S$, *(rectangular inequality), is a quasi metric and the pair* (S,G_d) *is quasi* G_d*-metric space.*

From (i), if $G_d(u,v,t) = G_d(u,t,v) = G_d(t,u,v) = G_d(t,v,u) = G_d(v,t,u) = G_d(v,u,t) = 0$, *then* $t = u = v$. *But* $u = v = t$ *does not assure that* $G_d(t,u,v) = 0$. *Along with (i) and (ii) if* $G_d(t,u,v) = G_d(t,v,u) = G_{d(u,t,v)} = G_d(u,v,t) = G_d(v,t,u) = G_d(v,u,t)$, *for all* $t,u,v \in S$ *holds then* (S,G_d) *becomes* G_{MLS}.

Example 2.6. *For a set* $S = R$ *and mapping* $G_d : S \times S \times S \longrightarrow [0,\infty)$ *defined by*

$$G_d(u,v,t) = d(u,v) + d(v,t) + d(u,t), \text{for all } u,v,t \in S$$

where $d : S \times S \longrightarrow [0,\infty)$ *is usual metric. Then clearly* (S,G_d) *is a genaralized metric Like Space.*

Definition 2.7. [1] $G_{MLS}(S,G_d)$ *is said to be complete if every Cauchy sequence in* (S,G_d) *is convergent in S.*

Definition 2.8. [1] *In* $G_{MLS}(S,G_d)$, *for* $t_0 \in S$, $0 < \daleth$, *closed ball* (C_B) *with center* t_0 *and radius* $\daleth$ *is*

$$C_B = \overline{B_{G_d}(t_0,\daleth)} = \{t \in S : G_d(t_0,t,t) \leq \daleth\}$$

Lemma 2.9. [9] *Let* (S,G_d) *be a* G_{MLS} *then for all* $t,u,v \in S$

$$G_d(t,t,u) \leq 2G_d(t,u,v)$$

Definition 2.10. [2] *Let* $\digamma_1$ *and* $\digamma_2$ *be self-mappings on S. If* $u = \digamma_1 v = \digamma_2 v$ *for some* $v \in S$, *then* $v \in S$ *is said to be a coincidence point for mappings* $\digamma_1$ *and* $\digamma_2$ *whereas* $v \in S$ *is called point of coincidence of mappings* $\digamma_1$ *and* $\digamma_2$.

Note that, if $u = v$ then $u \in S$ becomes common fixed point of self-mappings $\digamma_1$ and $\digamma_2$.

Proposition 2.11. [1] *Let* (S,G_d) *be a* G_{MLS}. *Then the function* $G_d(t,u,v)$ *is continuous for three variables.*

Proposition 2.12. [1] *In* G_{MLS} (S,G_d), *following conditions are equivalent:*

(i) $\{t_n\}$ *is Cauchy sequence,*

(ii) for $\varepsilon > 0$, *there exists* $n_0 \in \mathbb{N}$, *such that* $G_d(t_m,t_n,t_n) < \varepsilon$, *for all* $n,m \geq n_0$.

Definition 2.13. [1] *Let* (S,G_d) *be* G_{MLS} *and* $\{t_n\}$ *be a sequence in S. A point* $t \in S$ *is called a limit point of* $\{t_n\}$ *if* $\lim_{n\to+\infty} G_d(t,t_n,t_n) = 0$. *Also, one can say* $\{t_n\}$ *converges to t. Therefore, if* $t_n \to t$, *then for any* $\varepsilon > 0$, *there exists* $n_0 \in \mathbb{N}$, *such that* $G_d(t,t_n,t_n) < \varepsilon$, *for all* $n \geq n_0$.

Definition 2.14. [6] *Let (S, G_d) be a compact G_{MLS}, if and only if every sequence in S has convergent subsequence in S.*

Definition 2.15. [10] *Let (S, G_d) be a G_{MLS} then G_d is symmetric in two variables if for all $t, u \in S$*

$$G_d(t, u, u) = G_d(u, t, t).$$

3. Main Results

In this particular section, the article focuses on proving common fixed point results for self-mappings within the closed ball (C_B) rather than the entire space S. The context of the investigation is set within the framework of generalized metric-like spaces (G_{MLS}). Notably, the article also includes an example to validate the main result, which demonstrates that the contractive condition holds within C_B but fails within the broader space S. Specifically, it highlights the possibility that while the contractive condition may fail within the full space S, it can still be satisfied within subsets of S, such as the C_B. This example serves as concrete evidence to support the findings and reinforces the importance of considering the C_B when analyzing fixed point properties.

Theorem 3.1. *Suppose (S, G_d) be a symmetric and complete G_{MLS} and $\digamma_1, \digamma_2 : S \to S$ be any two dominated mappings. Suppose that,*

$$G_d(\digamma_1 t, \digamma_2 u, \digamma_2 v) \leq \xi G_d(t, u, v), \tag{3. 1}$$

for all $t, u, v \in C_B \subseteq S$ and for some $\xi \in [0, 1)$. Also,

$$G_d(t_0, t_1, t_1) = G_d(t_0, \digamma_2 t_0, \digamma_2 t_0) \leq (1 - \xi)\gimel. \tag{3. 2}$$

Then there exist unique $t_0 \in C_B$ such that $\digamma_1 t_0 = \digamma_2 t_0 = t_0$.

Proof. Let $t_0 \in C_B$ and conider the sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_{2n+1} = \digamma_2 t_{2n}$ and $t_{2n+2} = \digamma_1 t_{2n+1}$, where $\digamma_1$ and $\digamma_2$ are dominated mappings then $t_n \leq t_{n-1}$, for all $n = 0, 1, 2, \ldots$. If $t_{2n+1} = \digamma_1 t_{2n+1} = \digamma_2 t_{2n}$ then $t_{2n+1} \in C_B$ is fixed point of $\digamma_1$ and $\digamma_2$. But if $\digamma_1 t_{2n+1} \neq \digamma_2 t_{2n}$ then consider

$$\begin{aligned} G_d(t_0, t_1, t_1) &= G_d(t_0, \digamma_2 t_0, \digamma_2 t_0) \leq (1 - \xi)\gimel, \\ G_d(t_0, t_1, t_1) &\leq \gimel. \end{aligned}$$

Clearly $t_1 \in C_B$. Agnain, consider the relation

$$\begin{aligned} G_d(t_1, t_2, t_2) &= G_d(\digamma_2 t_0, \digamma_1 t_1, \digamma_1 t_1) = G_d(\digamma_1 t_1, \digamma_2 t_0, \digamma_2 t_0), \\ &\leq \xi G_d(t_0, t_1, t_1). \end{aligned}$$

Now for $t_2 \in S$, using triangular property,

$$\begin{aligned} G_d(t_0, t_2, t_2) &\leq G_d(t_0, t_1, t_1) + G_d(t_1, t_2, t_2), \\ &\leq (1 + \xi) G_d(t_0, t_1, t_1), \\ &\leq (1 + \xi)(1 - \xi)\gimel, \\ &\leq (1 - \xi^2)\gimel, \\ &\leq \gimel. \end{aligned}$$

Therefore, $t_2 \in C_B$. Again, consider the relation

$$\begin{aligned} G_d(t_2,t_3,t_3) &= G_d(F_1t_1,F_2t_2,F_2t_2),\\ &\leq \xi G_d(t_1,t_2,t_2),\\ &\leq \xi^2 G_d(t_0,t_1,t_1).\end{aligned}$$

For $t_3 \in S$

$$\begin{aligned} G_d(t_0,t_3,t_3) &\leq G_d(t_0,t_1,t_1)+G_d(t_1,t_2,t_2)+G_d(t_2,t_3,t_3),\\ &\leq G_d(t_0,t_1,t_1)+\xi G_d(t_0,t_1,t_1)+\xi^2 G_d(t_0,t_1,t_1),\\ &= (1+\xi+\xi^2)G_d(t_0,t_1,t_1),\\ &\leq (1+\xi+\xi^2)(1-\xi)\daleth, \because \text{(using (3.1))}\\ &= (1-\xi^3)\daleth,\\ &\leq \daleth.\end{aligned}$$

Therefore, $t_3 \in C_B$. Now let $t_4,t_5,t_6,\ldots,t_i \in C_B$, using mathematical induction for all $i = 0,2,4,6,\ldots$ gives,

$$\begin{aligned} G_d(t_i,t_{i+1},t_{i+1}) &= G_d(F_1t_{i-1},F_2t_i,F_2t_i),\\ &\leq \xi G_d(t_{i-1},t_i,t_i) = \xi G_d(F_2t_{i-2},F_1t_{i-1},F_1t_{i-1})\\ &= \xi G_d(F_1t_{i-1},F_2t_{i-2},F_2t_{i-2}),\\ &\leq \xi^2 G_d(t_{i-1},t_{i-2},t_{i-2}),\\ &\vdots\\ &\leq \xi^i G_d(t_0,t_1,t_1).\end{aligned}$$

Now by rectangular property for $t_i \in S$,

$$\begin{aligned} G_d(t_0,t_i,t_i) &\leq G_d(t_0,t_1,t_1)+G_d(t_1,t_2,t_2)+\ldots+G_d(t_{i-1},t_i,t_i),\\ &\leq G_d(t_0,t_1,t_1)+\xi G_d(t_0,t_1,t_1)+\ldots+\xi^{i-1}G_d(t_0,t_1,t_1),\\ &= (1+\xi+\xi^2+\ldots+\xi^{i-1})G_d(t_0,t_1,t_1),\\ &\leq (1-\xi^i)\daleth,\\ &\leq \daleth.\end{aligned}$$

Clearly $t_i \in C_B$, for all $i \in \mathbb{N}$. Hence, the sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq C_B$. For sequence $\{t_n\}_{n\in\mathbb{N}}$ to be Cauchy sequence, for $m,n \in \mathbb{N}$ with $n < m$, consider

$$\begin{aligned} G_d(t_n,t_m,t_m) &\leq G_d(t_n,t_{n+1},t_{n+1})+(t_{n+1},t_{n+2},t_{n+2})+\ldots+G_d(t_{m-1},t_{,m},t_m),\\ &\leq \xi^n(1+\xi+\xi^2+\ldots+\xi^{m-n-1})G_d(t_0,t_1,t_1),\\ &= \xi^n\left(\frac{1-\xi^{m-n}}{1-\xi}\right)G_d(t_0,t_1,t_1), \qquad (3.\ 3)\\ &\leq \frac{\xi^n}{1-\xi}G_d(t_0,t_1,t_1),\\ &\leq \frac{\xi^n}{1-\xi}(1-\xi)\daleth,\\ &\leq \xi^n\daleth.\end{aligned}$$

As $\xi \in [0,1)$, then $\xi^n \to 0$ if $n \to +\infty$. Therefore, for every $\varepsilon \in \mathbb{R}^+$, there exists $j \in \mathbb{R}$ such that,

$$G_d(t_n, t_m, t_m) \leq \xi^n] = \varepsilon, \text{ for } m > n \geq j.$$

Hence $\{t_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in C_B. As S is complete then there exists $t_0 \in S$ such that $t_n \to t_0$ for $n \to +\infty$ and hence

$$\lim_{n \longrightarrow +\infty} G_d(t_0, t_n, t_n) = 0.$$

To check $t_0 \in C_B$ is either common fixed point of $F_1, F_2 : S \to S$ or not consider

$$G_d(t_0, F_1 t_0, F_1 t_0) \leq G_d(t_0, t_{2n+1}, t_{2n+1}) + G_d(t_{2n+1}, F_1 t_0, F_1 t_0).$$

By using Definition 2.15 and Proposition 2.11, gives

$$\begin{aligned} G_d(t_0, F_1 t_0, F_1 t_0) &\leq G_d(t_0, t_{2n+1}, t_{2n+1}) + G_d(F_1 t_0, F_2 t_{2n}, F_2 t_{2n}), \\ &\leq G_d(t_0, t_{2n+1}, t_{2n+1}) + \zeta G_d(t_0, t_{2n}, t_{2n}), \\ G_d(t_0, F_1 t_0, F_1 t_0) &\to 0, \text{ when } n \to +\infty. \\ F_1 t_0 &= t_0. \end{aligned}$$

Again consider

$$\begin{aligned} G_d(t_0, F_2 t_0, F_2 t_0) &\leq G_d(t_0, t_{2n+2}, t_{2n+2}) + G_d(t_{2n+2}, F_2 t_0, F_2 t_0), \\ &\leq G_d(t_0, t_{2n+2}, t_{2n+2}) + \xi G_d(t_{2n+1}, t_0, t_0), \end{aligned}$$

By using Definition 2.15 and Proposition 2.11, gives

$$G_d(t_0, F_2 t_0, F_2 t_0) \leq 0, \text{ when } n \to +\infty.$$

As $G_d(t_0, F_2 t_0, F_2 t_0) \not< 0$, so

$$\begin{aligned} G_d(t_0, F_2 t_0, F_2 t_0) &= 0, \\ F_2 t_0 &= t_0. \end{aligned}$$

Hence $t_0 \in C_B$ is fixed point for mappings F_1 and F_2, i.e. $F_1 t_0 = F_2 t_0 = t_0$. for uniqueness of fixed point, consider $\{u_0, v_0\} \subseteq C_B$ such that $F_1(u_0) = F_2(u_0) = u_0$ and $F_1(v_0) = F_2(v_0) = v_0$. Now consider by relation,

$$\begin{aligned} G_d(F_1 u_0, F_2 v_0, F_2 v_0) &\leq \xi G_d(u_0, v_0, v_0), \\ G_d(u_0, v_0, v_0) &\leq \xi G_d(u_0, v_0, v_0), \\ (1-\xi) G_d(u_0, v_0, v_0) &\leq 0. \end{aligned}$$

As $1 - \xi > 0$, then

$$\begin{aligned} G_d(u_0, v_0, v_0) &= 0, \\ u_0 &= v_0. \end{aligned}$$

It is not true ($\because u_0 \neq v_0$). Therefore, common fixed point of F_1 and F_2, is unique. □

Remark 3.2. *Theorem 3.1, also holds for dominating mappings F_1 and F_2.*

Example 3.3. *Let $S = [0, \infty)$ and $G_d : S \times S \times S \to [0, \infty)$ be a mapping defined by,*

$$G_d(t, u, v) = \max\{|t|, |u|, |v|\}$$

for all $t, u, v \in S$, with order say $t \geq u \geq v$ then (S, G_d) is symmetric and complete G_{MLS}. Closed ball with $\daleth = 1$ and $t_0 = \frac{1}{2}$ is $C_B = [0, 1]$. Let $\digamma_1, \digamma_2 : S \to S$ are mappings defined by,

$$\begin{aligned} \digamma_1 t &= \begin{cases} \frac{w}{3} & \text{if } 0 \leq w \leq 1 \\ w + \frac{1}{2} & \text{if } 1 < w < \infty \end{cases}, \\ \digamma_2 t &= \begin{cases} \frac{w}{5} & \text{if } 0 \leq w \leq 1 \\ w + \frac{1}{3} & \text{if } 1 < w < \infty \end{cases}. \end{aligned}$$

Clearly $\digamma_1$ and $\digamma_2$ are, dominated mappings inside of C_B, but not dominated outside of C_B. Also for $\xi = \frac{1}{2} \in (0, 1)$ such that,

$$(1 - \xi)\daleth = \left(1 - \frac{1}{2}\right).1 = \frac{1}{2}$$

Also,

$$G_d(t_0, t_1, t_1) = \max\{|t_0|, |t_1|, |t_2|\} = \max\{|t_0|, |\digamma_2 t_0|, |\digamma_2 t_0|\}$$

As $\digamma_2$ being dominated mapping $\digamma_2 t_0 \leq t_0 \Rightarrow |t_1| \leq |t_0|$

$$\begin{aligned} G_d(t_0, t_1, t_1) &= |t_0| = \frac{1}{2}, \\ G_d(t_0, t_1, t_1) &= (1 - \xi)\daleth. \end{aligned}$$

Also for all $t, u, v \in [0, 1]$

$$G_d(\digamma_1 t, \digamma_2 u, \digamma_2 v) = G_d\left(\frac{t}{3}, \frac{u}{5}, \frac{v}{5}\right) = \max\left\{\left|\frac{t}{3}\right|, \left|\frac{u}{5}\right|, \left|\frac{v}{5}\right|\right\}.$$

As $\frac{t}{3} \geq \frac{t}{5} \geq \frac{u}{5} \geq \frac{v}{5}$, then

$$G_d(\digamma_1 t, \digamma_2 u, \digamma_2 v) = \left|\frac{t}{3}\right| = \frac{|t|}{3},$$

and

$$\begin{aligned} \xi G_d(t, u, v) &= \frac{1}{2}\max\{|t|, |u|, |v|\} = \frac{|t|}{2}, \\ G_d(\digamma_1 t, \digamma_2 u, \digamma_2 v) &< \xi G_d(t, u, v). \end{aligned}$$

Hence contractive condition is satisfied in C_B. For $t, u, v \in (1, \infty)$,

$$G_d(\digamma_1 t, \digamma_2 u, \digamma_2 v) = \max\left\{\left|t + \frac{1}{2}\right|, \left|u + \frac{1}{3}\right|, \left|v + \frac{1}{3}\right|\right\}$$

As $t+\frac{1}{2} \geq t+\frac{1}{3} \geq u+\frac{1}{3} \geq v+\frac{1}{3}$, *then*

$$G_d\left(\digamma_1 t, \digamma_2 u, \digamma_2 v\right)=\left|t+\frac{1}{2}\right|,$$

and

$$\begin{aligned} \xi G_d(t, u, v) &= \frac{1}{2} \max \{|t|,|u|,|v|\}=\frac{|t|}{2}, \\ G_d\left(\digamma_1 t, \digamma_2 u, \digamma_2 v\right) &\geq \xi G_d(t, u, v) . \end{aligned}$$

Clearly, outside of the C_B contractive condition is failed. This shows that all conditions of Theorem 3.1, for mappings $\digamma_1$ and $\digamma_2$ are satisfied inside of C_B. Moreover, there exists $0 \in [0,1]$ *such that* $\digamma_1 0 = \digamma_2 0 = 0$.

Note that, Theorem 3.1, can also be proved for single mapping as shown in following corollary.

Corollary 3.4. *Suppose (S, G_d) be a symmetric and compact G_{MLS} and $\digamma_1 : S \to S$ be any dominated mapping and $t_0, t, u, v \in S$, $\gimel > 0$. Suppose that, there exist $\xi \in [0,1)$ such that,*

$$G_d(\digamma_1 t, \digamma_1 v, \digamma_1 v) \leq \xi G_d(t, u, v)$$

for all $t, u, v \in C_B \subseteq S$, and

$$G_d(t_0, t_1, t_1) = G_d(t_0, \digamma_1 t_0, \digamma_1 t_0) \leq (1-\xi)\gimel.$$

Then there exist unique $t_0 \in C_B$ such that $\digamma_1 t_0 = t_0$.

Proof. (S, G_d) being compact metric space is complete and totally bounded. Then follow the proof of Theorem 3.1. □

Corollary 3.5. *Suppose (S, G_d) be a symmetric and complete G_{MLS} and $\digamma_1 : S \to S$ be two dominated mapping and $t_0, t, u, v \in S$, $\gimel > 0$. Suppose that there exists $\xi \in [0,1)$ such that,*

$$G_d(\digamma_1 t, \digamma_1 v, \digamma_1 v) \leq \xi G_d(t, u, v)$$

for all $t, u, v \in C_B \subseteq S$, and

$$G_d(t_0, t_1, t_1) = G_d(t_0, \digamma_1 t_0, \digamma_1 t_0) \leq (1-\xi)\gimel$$

Then there exist unique $t_0 \in C_B$ such that $\digamma_1 t_0 = t_0$.

Proof. In Theorem 3.1 using $\digamma_1 = \digamma_2$ to obtain unique fixed point for single mapping $t_0 = \digamma_1 t_0$. □

Proof of the following corollary for noncompareable elements, is same as the proof of Theorem 3.1.

Corollary 3.6. *Suppose (S, G_d) be a symmetric and complete G_{MLS} and $\digamma_1, \digamma_2 : S \to S$ are any two dominated mappings and $t_0, t, u, v \in S$, $\gimel > 0$. Suppose that there exists $\xi \in [0,1)$ such that,*

$$G_d(\digamma_1 t, \digamma_2 u, \digamma_2 v) \leq \xi G_d(t, u, v),$$

for all $t, u, v \in C_B$, and

$$G_d(t_0, t_1, t_1) \leq (1-\xi)\gimel.$$

Then there exist unique $t_0 \in C_B$ such that $\digamma_1 t_0 = \digamma_2 t_0 = t_0$.

Proof. Consequence of Theorem 3.1. □

Corollary 3.7. *From Theorem 3.1, iterative sequence $\{t_n\}$, with arbitrary $t_0 \in C_B \subseteq S$, converges to unique common fixed-point $t_0 \in C_B$ of dominated mappings $\digamma_1$ and $\digamma_2$. Error estimates are the prior estimate*

$$G_d(t_n, t_0, t_0) \leq \frac{\xi^n}{1-\xi} G_d(t_0, t_1, t_1) \tag{3. 4}$$

and the posterior estimate

$$G_d(t_n, t_0, t_0) \leq \frac{\xi}{1-\xi} G_d(t_{n-1}, t_n, t_n) \tag{3. 5}$$

Proof. From (3. 3),

$$G_d(t_n, t_m, t_m) \leq \xi^n \left(\frac{1-\xi^{m-n}}{1-\xi} \right) G_d(t_0, t_1, t_1)$$

As the sequence $\{t_n\}$ is convergent to $t_0 \in C_B$, then $t_m \to t_0$ as $m \to \infty$ and $1-\xi^{m-n} \to 1$. Therefore, above relation leads to the prior estimate, i.e.,

$$G_d(t_n, t_0, t_0) \leq \frac{\xi^n}{1-\xi} G_d(t_0, t_1, t_1)$$

Setting $n = 1$ and write u_0 for t_0 and u_1 for t_1 in above relation to get

$$G_d(u_1, t_0, t_0) \leq \frac{\xi}{1-\xi} G_d(u_0, u_1, u_1)$$

Letting $u_0 = t_{n-1}$ then $u_1 = \digamma_2 u_0 = \digamma_2 t_{n-1} = t_n$ in above relation gives the posterior estimate (3. 3),

$$G_d(t_n, t_0, t_0) \leq \frac{\xi}{1-\xi} G_d(t_{n-1}, t_n, t_n)$$

Note that, the prior error bound (3. 4) can be used in the beginning of calculation for assessing the necessary number of steps to get required accuracy. While posterior error bound (3. 5) can be used at middle of the road stages or at the end of calculation. Posterior error bound at any rate as exact as prior error bound. □

Example 3.8. *Let $S = [0, \infty)$ and $G_d : S \times S \times S \to [0, \infty)$ be a mapping defined by,*

$$G_d(t, u, v) = \max\{|t|, |u|, |v|\}$$

for all $t, u, v \in S$ with order say $t \geq u \geq v$ then (S, G_d) is symmetric and complete G_{MLS}. Closed ball with $\gimel = 1$ and $t_0 = \frac{1}{2}$ is $C_B = [0, 1] \subset S$. Let $\digamma_1, \digamma_2 : S \to S$ are mappings defined by,

$$\digamma_2 t = \begin{cases} \dfrac{t}{5} & \text{if } 0 \leq t \leq 1 \\ t + \dfrac{1}{3} & \text{if } 1 < t < \infty \end{cases},$$

$$\digamma_1 t = \begin{cases} \dfrac{t}{3} & \text{if } 0 \leq t \leq 1 \\ t + \dfrac{1}{2} & \text{if } 1 < t < \infty \end{cases}.$$

Obviouslv, F_1 and F_2 are, dominated mappings inside of C_B but not dominated outside of C_B. Also let $\xi = \frac{1}{2} \in [0,1)$. Construct the iterative sequence with initial guess $t_0 = \dfrac{1}{2} \in [0,1]$ as

n	$t_{2n+1} = F_2 t_{2n}$	$t_{2n+2} = F_1 t_{2n+1}$
0	$\frac{3}{2(15)^1}$	$\frac{1}{2(15)^1}$
1	$\frac{3}{2(15)^2}$	$\frac{1}{2(15)^2}$
2	$\frac{3}{2(15)^3}$	$\frac{1}{2(15)^3}$
$\vdots$	$\vdots$	$\vdots$
m	$\frac{3}{2(15)^m}$	$\frac{1}{2(15)^m}$.

Let $2m-1=\wp$, then for every odd $\wp \in \mathbb{N}$ gives,

$$t_\wp = \frac{3}{2\left(\sqrt{15}\right)^{\wp+1}},\; t_{\wp+1} = \frac{1}{2\left(\sqrt{15}\right)^{\wp+1}} \tag{3. 6}$$

Now as,

$$G_d(t_0,t_1,t_1) = \max\left\{|t_0|,|t_1|\right\} = t_0 = \frac{1}{2}.$$

Also, as

$$G_d(t_\wp,t,t) = \max\left\{|t_\wp|,|t|\right\}.$$

As from (3. 6), sequence $\{t_\wp\} \in C_B$ is decreasing and bounded below. Then there exist greatest lower bound $t_0 \in [0,1]$ such that $t_\wp \geq t_0$ for all $\wp \in \mathbb{N}$ and hence,

$$G_d(t_\wp,t_0,t_0) = t_\wp.$$

Now from Corollary 3.7 (3. 4),

$$\begin{aligned} G_d(t_\wp,t_0,t_0) &\leq \frac{\xi^\wp}{1-\xi} G_d(t_0,t_1,t_1), \\ t_\wp &\leq \frac{\left(\frac{1}{2}\right)^\wp}{1-\frac{1}{2}}.\frac{1}{2}, \\ \frac{3}{2\left(\sqrt{15}\right)^{\wp+1}} &\leq \frac{1}{2^\wp}, \\ \frac{3}{2\sqrt{15}} &\leq \frac{\left(\sqrt{15}\right)^\wp}{2^\wp}, \\ \frac{\ln\frac{3}{2\sqrt{15}}}{\ln\frac{\sqrt{15}}{2}} &\leq \wp, \\ -1.4353 &\leq \wp. \end{aligned}$$

For every odd integer $\wp \geq 1$, sequence $\{t_\wp\}$, becomes convergent, i.e., for every odd $\wp \in \mathbb{N}$ relation (3. 6) *becomes,*

$\wp$	$t_\wp$	$t_{\wp+1}$
1	0.10000000000	0.03333333333
3	0.00666666667	0.00222222222
5	0.00044444444	0.00014814815
7	0.00002962296	0.00000987654
$\vdots$	$\vdots$	$\vdots$

Hence, from above table, when odd $\wp \in \mathbb{N}$ increases then $t_\wp \to t_0$ and $t_0 \in [0,1]$ becomes common fixed point of mapping $\digamma_1$ and $\digamma_2$. Clearly, from the iterative sequence $t_0 = 0 \in [0,1]$, such that $\digamma_1 0 = \digamma_2 0 = 0$.

4. Application To Nonlinear System of Integral Equations

In this section, the nonlinear integral equation is resolved by employing Theorem 3.1. To accomplish this, the interval $S = ([t_1, t_2], \mathbb{R}^n)$, where $C_B = [t_1, t_2]$ and $b > 0$. By utilizing Theorem 3.1, a suitable solution for the nonlinear integral equation within the specified interval S can be obtained by using

$$G_d(u, v, v) = 2 \sup_{\tau \in [t_1, t_2]} |u(\tau) - v(\tau)| \sqrt{1 + b^2} e^{i \tan^{-1} b} \tag{4. 7}$$

for all $u, v \in S$, then (S, G_d) is complete G_{MLS}. The following result establishes an existence theorem for a common solution to a system of two nonlinear Urysohn integral equations.

Theorem 4.1. *Let's consider the Urysohn integral equations given by*

$$\begin{cases} u(t) = \int_{t_1}^{t_2} L_1(t, s, u(s))\, ds + g_1(t), \\ u(t) = \int_{t_1}^{t_2} L_2(t, s, u(s))\, ds + g_2(t), \end{cases} \tag{4. 8}$$

where $t \in C_B$, and $u, g_1, g_2 \in S$. The functions $L_1, L_2 \in [t_1, t_2] \times [t_1, t_2] \times \mathbb{R}^n \to \mathbb{R}^n$. Suppose that for each $u, v \in S$, $M_u, N_u \in S$, defined as

$$M_u(t) = \int_{t_1}^{t_2} L_1(t, s, u(s))\, ds, \quad \textit{and} \quad N_u(t) = \int_{t_1}^{t_2} L_2(t, s, u(s))\, ds,$$

for all $t \in [t_1, t_2]$. If there exists $\pounds \in [0, 1)$ such that for every $u, v \in S$ for all $t \in [t_1, t_2]$. If there exists a value $\pounds \in [0, 1)$ such that for every $u, v \in S$, the following inequality holds

$$|M_u(t) - N_v(t) + g_1(t) - g_2(t)| \sqrt{1 + b^2} e^{i \tan^{-1} b} \leq \pounds G_d(u, v, v), \tag{4. 9}$$

then the system of integral equations (4. 8) *has a unique common solution in S.*

Proof. Define $\digamma_1, \digamma_2 \in S$

$$\digamma_1 u = M_u + g_1, \quad \digamma_2 u = N_u + g_2. \tag{4. 10}$$

Then, from (4. 7)

$$\begin{aligned} G_d\left(F_1 u, F_2 v, F_2 v\right) &= \sup_{t\in[t_1,t_2]} \left|F_1 u\left(t\right) - F_2 v\left(t\right)\right|, \\ &= \sup_{t\in[t_1,t_2]} \left|M_u\left(t\right) - N_v\left(t\right) + g_1\left(t\right) - g_2\left(t\right)\right| \sqrt{1+b^2} e^{i\tan^{-1} b}, \\ &\leq \pounds G_d\left(u, v, v\right), \text{by } (x), \end{aligned}$$

for every $u, v \in S$. By Theorem 3.1, self-mapping $F_1, F_2 \in S$ have usinque common fixed point in S, which is the unique solution of nonlinear system of Urysohn integral equations (4. 8). □

5. Conclusion

The opening section of this article provides a brief introduction, while the subsequent section focuses on laying the groundwork and providing necessary background information to aid readers in comprehending the topic. The third section of the article is dedicated to presenting the main results, specifically addressing the existence of a common fixed point for two locally contractive mappings within G_{MLS}. The validity of Theorem 3.1 is supported by an accompanying example. Additionally, within this section, Corollary 3.6 is discussed, which covers both prior and posterior error estimates. Furthermore, Corollary 3.6 is employed to approximate the common fixed point of the locally contractive mappings in G_{MLS}. In the final section, the article examines the existence of a common solution to a system of two nonlinear Urysohn integral equations.

References

[1] M. Abbas and S. Z. Nemeth, *Finding solutions of implicit complementarity problems by isotonicity of metric projection,* Nonlinear Analysis, No. 75, (2012) 2349-2361.

[2] R. Agarwal and E. Karapinar, *Remarks on some coupled fixed-point theorems in G-metric spaces,* Fixed Point Theory and Applications, (2013) 1-33.

[3] I. K. Agwu and D. I. Igbokwe, *A Modified Proximal Point Algorithm for Finite Families of Minimization Problems and Fixed Point Problems of Asymptotically Quasi-nonexpansive Multivalued Mappings,* Punjab Univ. J. math. **54,** No. 8 (2022),495-522.

[4] M. Arshad, *Fixed Points of a pair of dominating mappings on a closed ball in ordered partial metric spaces,* Journal of Advanced Mathematical Studies **2,** No. 7 (2014) 123-134.

[5] S. Banach, *Sur les operations dans les ensembles abstraits et leur applications aux equations integrals,* Fundam Math. **3**, 133-181.

[6] N. L. Carothers, *Real Analysis*, Cambridge University Press, UK, 2012.

[7] B. C. Dhage, *Generalized metric spaces mappings with fixed point,* Bull. Calcutta Math. Soc., No. 84 (1992) 1-336

[8] M. Gugnani, M. Aggarwal and R. Chugh, *Common fixed point results in G-metric spaces and applications,* International Journal of Computer Applications **43**, No. 11 (2012) 38-42.

[9] E. Karanpinar and Ravi P. Agarwal, F*urther fixed-point results on G-metric spaces,* Fixed Point Theorem and Applications, (2013) 1-14.

[10] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces,* Journal of Nonlinear and Convex Analysis **2**, No. 7 (1992) 289-297.

[11] J. J. Nieto and R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations,* Springer **3**, No. 22 (2005) 223- 239.

[12] M. N. Nikhate, S. B. Kiwne and K. L. Bondar, *A Related Fixed Point Theorems using Contractive Mapping on Six Metric Spaces,* Punjab Univ. J. math. **52**, No. 2 (2020), 1-13.

[13] A. C. M. Ran and M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations,* Proceedings of the American Mathematical Society **132,** No. 5 (2004) 1435-1443.

[14] N. S. Rao, K. Kalyani and K. Prasad, Coincidence Point Results in Ordered Metric Spaces and its Applica-tion, Punjab Univ. J. math. **54**, No. 1 (2022),33-43.

[15] A. Shoaib, S. Mustafa and A. Shahzad, Common Fixed Point of Multivalued Mappings in Ordered Dislocated Quasi G-Metric Spaces, Punjab Univ. J. math. **52**, No. 10 (2020), 1-23.

[16] A.Shoaib,Z. Nisar, A. H ussain, ¨O. ¨Ozer and M. Arshad, *Modified Banach Fixed Point Results for Locally Contractive Mappings in Complete Gd -Metric Like Space*, Electronic Journal of Mathematical Analysis and Applications **1**, No. 6 (2018) 144-155