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Approximation of Nonlinear Sine-Gordon Equation via RBF-FD Meshless Approach

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Abstract.: Partial differential equations (PDEs) describe simulation of physical phenomena occurring in different fields of science and engineering. The analysis of solutions of nonlinear wave equations have been gaining a lot of popularity in the last two decades. Such wave equations have many applications in applied mathematics and theoretical physics. The importance of PDEs that explain nonlinear waves defined by Sine-Gordon (SG) equation are crucial. The SG equation is a particular instance of the Klein-Gordon (KG) equation, which is crucial in a number of scientific fields, such as solid state physics, nonlinear optics, and quantum field theory. This equation is also a description of a soliton wave that exists in many physical situations. Different analytical as well as numerical techniques were used to develop the exact and approximate solution of SG equation. In this article, we explore the numerical solution of the one-dimensional nonlinear SG problem using RBF-FD approach. The scheme is a combination of radial basis functions (RBFs) with finite differences (FD) for constructing local spatial approximations to SG equation. For execution of time variable in the given model equation, Runge-Kutta (RK) time steeping approach is utilized. To verify the validity of our method, solutions to some test problems are examined. Accuracy of the proposed scheme is verified through L_∞, L_2, "RMS" and "MAE" error norms. The solutions acquired by suggested RBF-FD approach are also compared to earlier work and the obtained results are batter and in good agreement with the exact solution.

AMS (MOS) Subject Classification Codes: 35N05; 35B10; 65M70; 65D12; 65M12; 65M06; 65J08
Key Words: Partial Differential Equations (PDEs), Sine-Gordon (SG) Equation, Meshfree Approache, RBFs Methods, FD Method, RBF-FD Method.

1. Introduction

Indeed, Nonlinear partial differential equations (NPDEs) are a significant tool for the analysis of nonlinear physical processes and natural phenomena such as in ocean engineering, physics, fluid mechanics, geochemistry, plasma physics, optical fibers, geophysics, and many other scientific areas [39, 69, 44]. Nonlinear processes are a field of interest to researchers in modern times. They have focused on finding the analytical or exact solutions to problems due to their contribution to the analysis of the actual system characteristics. Many disciplines of applied mathematics and physics produce these (NPDEs) wave equations as one-dimensional classical field theories. Solutions that behave in a particle-like manner are especially significant, both practically and theoretically. A pulse solution with velocity and position that is not zero is referred to as a particle-like solution. Surprisingly, the extraordinary particle-like stability of these pulse solutions are also visible. When two or more pulses collide, a composite pulse is created. This composite pulse immediately disintegrates into its component pulses, each of which regains its original shape and velocity. This type of particle-like solution is referred to as a ”soliton”. Helal [37] carried out a comprehensive overview of soliton solutions for some common PDEs and introduced various analytical and numerical treatment techniques. According to [37], solitons represent a special type of nonlinear localized wave. In fact, a ”soliton” describes any solution of a nonlinear equation or system that: (i) represents a permanent wave; (ii) is localized, decaying or becoming constant at infinity, and (iii) may interact strongly with other solitons such that after the interaction it retains its form, almost obeying the principle of superposition [37]. The theory of solitons and nonlinear evolution equations (NLEEs) has made a lot of progress in the last couple of decades and we encourage readers who are interested to read excellent overall summary of the subject [57, 22, 73, 1, 43, 42]. The nonlinear SG equation appears in a wide range of applications, including fluxion propagation in Josephson junctions [41], nonlinear physics, fluid motion stability, dislocations in crystals, differential geometry, a stiff pendulum swinging back and forth on a stretched wire, nonlinear optics, solid state physics and applied sciences [2, 37, 21]. The SG equation is a particular instance of KG equation, which is crucial in many scientific applications, such as solid state physics, quantum field theory, and nonlinear optics [5, 23, 12, 20]. This equation is also a description of a soliton wave. Numerous efforts have been made to develop numerical methods that are effective for wave simulation performance, and this has led to a continuing desire to increase the accuracy of these equations as well as the analytical studies. Additionally, the finite-difference (FDM), finite-element (FEM), pseudo-spectral (PS), and Adomian decomposition methods (ADM) were used to numerically solve the SG equation [18, 20, 66, 2, 6].

Several mesh-based approaches (such as FDM in 1950s [70, 45], FEM in 1960s [65, 71], and Spectral in 1970s [9, 10]) and meshfree approaches (similar to Radial Basis Functions (RBFs) in 1980s [35, 36]) have been developed for approximation of ODEs and PDEs. Meshfree (RBFs) approaches have a high convergence rate, stable, accurate, and infinitely smooth in complex geometry. The method most frequently used in the field of theory of multivariate approximations is the RBFs meshfree approach. A more generalized version of MQ approach is RBF method. Rolland L. Hardy created the MQ RBF, which has received substantial theoretical investigation and application [35, 36]. Following Hardy's discovery, researchers in a range of domains began utilizing the MQ technique. The MQ technique is extremely beneficial in domains of geophysics, geodesy, and other sciences [36]. RBFs are also used in numerical techniques to precisely solve multidimensional PDEs in a variety of applied science areas, as an example see [7, 8, 24, 26, 27, 68]. RBFs are frequently classified into three types: (i) RBFs with global support and infinite smoothness (GSIS) with a free parameter ζ that has spectral convergence rates, (ii) RBFs with global support and finite smoothness (GSFS) and (iii) RBFs with compact support and finite smoothness (CSFS), algebraic convergence rates exist for both of later. The following table provide a few of the well-known radial basis functions oftenly used in literature.

Where ζ, an open parameter that has a real value greater than zero. Yet a detailed investigation of the choice of best value for this parameter is currently ongoing. For researchers, it continues to be a challenge (view the citations [35, 32, 29, 49, 63, 25, 56, 67]). The shape parameter is numerically determined in order to regulate function shape, solution correctness, and conditioning of system matrix. Several scholars have investigated the shape parameter (for instance see [61, 16, 11, 47]). The shape parameter is situation-dependent, according to [11], that is the behaviour of the approximate function is an important factor to take into account when choosing the appropriate value for the shape parameter. Madych [47] has shown that the accuracy of

Name and Type	RBFs	$\Phi(r)$
GSIS	GMQ	$(1+\zeta^2r^2)^\lambda$, $\lambda>0, \lambda\notin\mathbb{N}$
	GA	$e^{-\zeta^2r^2}$
	GIMQ	$(1+\zeta^2r^2)^{-\lambda}$, $\lambda>0, \lambda\notin\mathbb{N}$
CSFS	C_1^4	$(1-\zeta r)_+^5(1+5\zeta r+8\zeta^2r^2)$
	C_3^4	$(1-\zeta r)_+^6(35\zeta^2r^2+18\zeta r+3)$
	C_5^4	$(1-\zeta r)_+^7(16\zeta^2r^2+7\zeta r+1)$
GSFS	TPS	r^2log(r)
	MN	$r^{2\lambda-1}$, $\lambda\in\mathbb{N}$
	PH	$r^{2\lambda}$log(r), $\lambda\in\mathbb{N}$

the RBF interpolant can be greatly increased by increasing the value of ζ. The authors [38, 33, 34] employed the cross validation method to establish the ideal value for the shape parameter. A Contour-Pad approach and RBF-QR algorithm were developed by [31] to overcome the ill-conditioned state of the sphere's surface caused by the RBFs interpolation. we employed the following algorithm in our computation, related to the theory of local RBF-FD interpolation for selecting a suitable shape parameter value.

Algo:
• Kept approximately the condition number in the range $10^{12}<\kappa<10^{16}$ for our problem system matrices.
• Decompose the interpolation matrix as $[L,S,U]=SVD(B_i)$. Here SVD is the singular value decomposition of the interpolation matrix B_i of order $n\times n$ corresponding to each local sub-domain Ω_i, and S is the diagonal matrix having n singular values of B_i, and $\kappa=\|B_i\|\|B_i\|^{-1}=max(S)/min(S)$, denotes condition number of matrix B_i.
• Search for ζ until κ satisfy the condition $10^{12}<\kappa<10^{16}$, using the algorithm below.

Algorithm:
$\kappa=1$,
$10^{12}<\kappa<10^{16}$,
while $\kappa<\kappa_{min}$ and $\kappa>\kappa_{max}$,
L,S,U=SVD(B),
$\kappa=\frac{max(B)}{min(B)}$,
if $\kappa<\kappa_{min}$, $\zeta=\zeta-\Delta\zeta$,
if $\kappa>\kappa_{max}$, $\zeta=\zeta+\Delta\zeta$,
ζ(optimal)=ζ.
When the above condition is satisfied a good value of ζ is obtained, the inverse is computed using, $B^{-1}=(LSU^T)^{-1}=US^{-1}L$.

2. RBF-FD APPROACH

Since RBFs methods possessed considerable attention in scientific community as a truly mesh-free methods and of their ability to achieve spectral accuracy for the PDEs solutions on irregular domain. Besides the competitive accuracy and convergence contrast to other state-of-arts methods, they also enjoy large time step stability [7, 26, 4]. To overcome some of the drawbacks and difficulties (like resulting linear system ill-conditioning) of RBFs methods, Several authors independently proposed a local version of the method simultaneously, which remains always an important alternative. The intention of the local method is the offering of spectral accuracy intrinsic to the global method to get a better-conditioned sparse linear system having the ability to solve large dimensional PDEs. Further benefit of local version methods is their appropriateness for problems having discontinuous boundary conditions. A local version of the method called as Local Radial Basis Functions Collocation Method (LRBFCM) described by Chen [14]. In this approach, instead of using all the nodes in whole domain, only the local approximation is to be considered for collocation [46, 52, 13]. Another very promising local approach known as RBF-FD method, which is the combination of features of RBF with conventional finite differences (FD) to get best out of RBFs (achieving high accuracy) on scattered nodes without requiring a computational mesh. Tolstykh firstly introduced RBF-FD method in 2000 and then Wright [62, 75]. In addition, the convergence analysis of the above mentioned RBF-FD method has been analytically proved by Bayona, and Moscoso et al., [3]. This method has been applied successfully to a great variety of problems in the last years, see for example [13, 19, 30, 46, 50, 51, 53, 58, 59, 72, 40, 68].

The following is the order in which the paper is organized. Section 3 provides specifics of the equations under consideration, and Sections 4 and 5 outline the suggested scheme and its stability. In Section 6, numerical examples and outcomes are provided. Section 7 includes comments and closing statements.

3. Governing equations

The primary equation of the aforementioned phenomena alongwith initial and Dirichlet boundary condition, stated in the classical order, is as follows:

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) - sin(u(x,t)),\ (x,t) \in [a,b] \times [0,T], \\ u(x,0) = u_1(x),\ u_t(x,0) = u_2(x),\ x \in [a,b], \\ u(a,t) = h_1(t),\ u(b,t) = h_2(t),\ t \in (0,T]. \end{cases} \tag{3. 1}$$

The transformed form of the above SG equation into coupling equations is given by;

$$\begin{cases} u_t(x,t) = v(x,t),\ v_t(x,t) = u_{xx}(x,t) - sin(u(x,t)), \\ u(x,0) = f(x),\ v(x,0) = g(x),\ x \in [a,b], \\ u(a,t) = f_1(t),\ u(b,t) = f_2(t),\ t \in (0,T], \\ v(a,t) = g_1(t),\ v(b,t) = g_2(t),\ t \in (0,T]. \end{cases} \tag{3. 2}$$

4. Description of RBF-Finite Differences method

4.1. Classical finite difference method. Let us look at a classical finite difference method for approximating the derivative of function $u(x,y)$ with respect to x. Let the derivative at any grid point (i,j) of rectangular grid be written as

$$\frac{\partial u}{\partial x}\ |_{(i,j)} \approx \sum_{k\in\{i-1,i,i+1\}} w_{(k,j)} u_{(k,j)}, \tag{4. 3}$$

where $u_{(k,j)}$ is the function value at the grid point (k,j), the unknown coefficients $w_{(k,j)}$ are computed using polynomial interpolation or Taylor series, while the set of nodes $\{(i-1,j),(i,j),(i+1,j)\}$ is a stencil in the finite difference literature. This approach becomes very restricted in high dimensions because it is possible only for some types of structured nodes. It severely limits the geometric flexibility of the method [15]. The methodology for computing the coefficients of finite difference formulas for any scattered points with dimensions of more than one has the problem of well-posedness for polynomial interpolation [76]. Thus, the combination of RBFs and finite difference methodology (RBF-FD) is introduced to overcome the well-posedness problem. Local RBFs, also known as RBF-FD have produced a lot of interest due to their interpolation and differentiation matrix structure. The interpolation and differentiation matrices generated by local RBFs have a controllable degree of sparsity. It has been found in some situations that the local RBF can match or provide accuracy similar to that of the global RBF method, for smaller grid sizes [17]. It uses only some nodes surrounding x_i, called a stencil, to approximate the derivatives of function $u(x)$ at x_i.

4.2. RBF-FD method for time-dependent PDE. Now we present an outline of RBFs along with finite-difference (RBF-FD) formulation for time dependent PDEs solution. Consider the following general time dependent PDE of the frame

$$u_t(x,t) = \mathcal{L}u(x,t),\ \text{such that}\ x \in \Omega \subseteq \mathbb{R}^d,\ d \geq 1, t > 0, \tag{4. 4}$$

associated with the following initial and boundary conditions

$$u(x,0) = u_0(x),\ x \in \Omega,\ \mathcal{B}u(x,t) = h(x,t),\ x \in \partial\Omega, \tag{4. 5}$$

where u_0 and h are certain provided functions, while the spatial operators $\mathcal{L}$, $\mathcal{B}$ representing the differential operators.

Suppose $\Omega = \{x_1, ..., x_N\}$ be a partition of the domain and $S^I = \{x_1^{(i)}, ..., x_{N_I}^{(i)}\} \subseteq \Omega$ be a stencil corresponding to x_i including N_I nodes so that $x_i \in S^I$ and $N_I \leq N$. So, for linear partial differential operator $\mathcal{L}$ we can use the local approximant

$$\mathcal{L}u(x_i) \simeq \sum_{j=1}^{N_I} w_j^{(i)} u(x_j^{(i)}). \tag{4. 6}$$

The RBF-FD approach overcomes this difficulty for calculating the unknown weights $\{w_j^{(i)}\}_{j=1}^{N_I}$ using the idea of determining these weights by imposing the requirement that the linear combination (4. 6) must be exact for RBFs, $\{\phi_j(x, c)\}_{j=1}^{N_I}$, centered at each of the node locations of stencil S^I [76], so that

$$\mathcal{L}\phi_k(x_i, c) = \sum_{j=1}^{N_I} w_j^{(i)} \phi_j(x_k, c),\ k = 1, ..., N_I. \tag{4. 7}$$

It concludes to the following algebraic linear system

$$\mathit{\Phi}\mathbf{w^I} = [\mathcal{L}\mathit{\Phi}]^I, \tag{4. 8}$$

in which $\mathit{\Phi}$ is the $N_I \times N_I$ interpolant matrix with elements

$$\phi_{kj} = \phi_j(x_k, c),\ k, j = 1, ..., N_I, \tag{4. 9}$$

$\mathbf{w^I}$ is the $N_I \times 1$ coefficient vector including the weight coefficients $\{w_j^{(i)}\}_{j=1}^{N_I}$, named RBF-FD coefficients, and $[\mathcal{L}\mathit{\Phi}]^I$ is the $N_I \times 1$ right-hand side vector containing the values $\mathcal{L}\phi_k(x_i, c)$ for $k = 1, ..., N_I$. Due to the non singularity of the interpolant matrix $\mathit{\Phi}$ (see [74, 8, 55]), one obtains the weights vector $\mathbf{w}^I$

$$\mathbf{w}^I = \mathit{\Phi}^{-1}[\mathcal{L}\mathit{\Phi}]^I. \tag{4. 10}$$

In fact, the RBF-FD method completely works like the same FD method except in the way of calculating the weights $\{w_j^{(i)}\}_{j=1}^{N_I}$, hence it can be considered as an enhanced FD method. Similar procedure can be used for the boundary operator $\mathcal{B}$. Finally, the discretization for problem (4. 4)-(4. 5) can be written as

$$\acute{u} = \mathbf{M}(u). \tag{4. 11}$$

When $\mathbf{M}$ is a $N \times N$ dimensional sparse differentiation matrix, every row of the $\mathbf{M}$ has entries with $N - n$ zeros and n non-zero values. The number n represents the number of stencil components. Similarly we can easily produced discretization of boundary operator $\mathcal{B}$ as $\mathcal{L}$. The system of ODEs (4. 11) can now be solved with any ODE solver like ode45, ode113, and ode23 etc from Matlab. Each effective ODE solver will select a suitable length (step) of time T to fix the stiffness of ODE system (4. 11) in time see [64]. In this study we have used classical fourth order Runge-Kutta (RK-4) method.

5. Stability of the proposed numerical schemes

In our proposed numerical scheme which is based on RBF-FD method we have transformed the time-dependent partial differential equation into an ODEs system in time see equation (4. 11). This type of technique is called the method of lines by which we can solve this system of coupled ODEs using the finite difference method in time for example RK methods, etc. The method of lines stability may be estimated by the well known rule of thumb. It is shown in the work of [64], that the method of lines will be stable, when the eigenvalues of spatial discretization operator, linearized and scaled by step size δt, lie in region of stability of the corresponding time-discretization operator. The stability region is a part of a multifaceted plane (complex plane) entailing of those eigenvalues for which the schemes construct a bounded solution. The stability of equation (4. 11) depends on the eigenvalues of the coefficient matrix. Hence, to show the stability of the numerical solution of (4. 4)-(4. 5), it is satisfactory to display that the real term of every eigenvalue $Re(\lambda_i)$ of the matrix $\mathbf{M}$ is non-positive, i.e., $Re(\lambda_i) \leq 0$ for all $i = 1, 2, ...n$, for more details, see [60]. Notice that the traditional RK method of order four stability criteria is $(-2.78 < \lambda\delta t < 0\ \forall\ \lambda)$. For more details on stability of RBF method for time dependent PDEs readers are refer to see for example [28, 48, 54]. Here in this study, it is shown that the current RBF-FD (localized) numerical scheme is unconditionally stable for all values of RBFs shape parameter and small step size δt, when solving the proposed SG model equation. To examine the stable and unstable eigenvalue spectrum we have calculated the eigenvalues of the matrix $\mathbf{M}$,

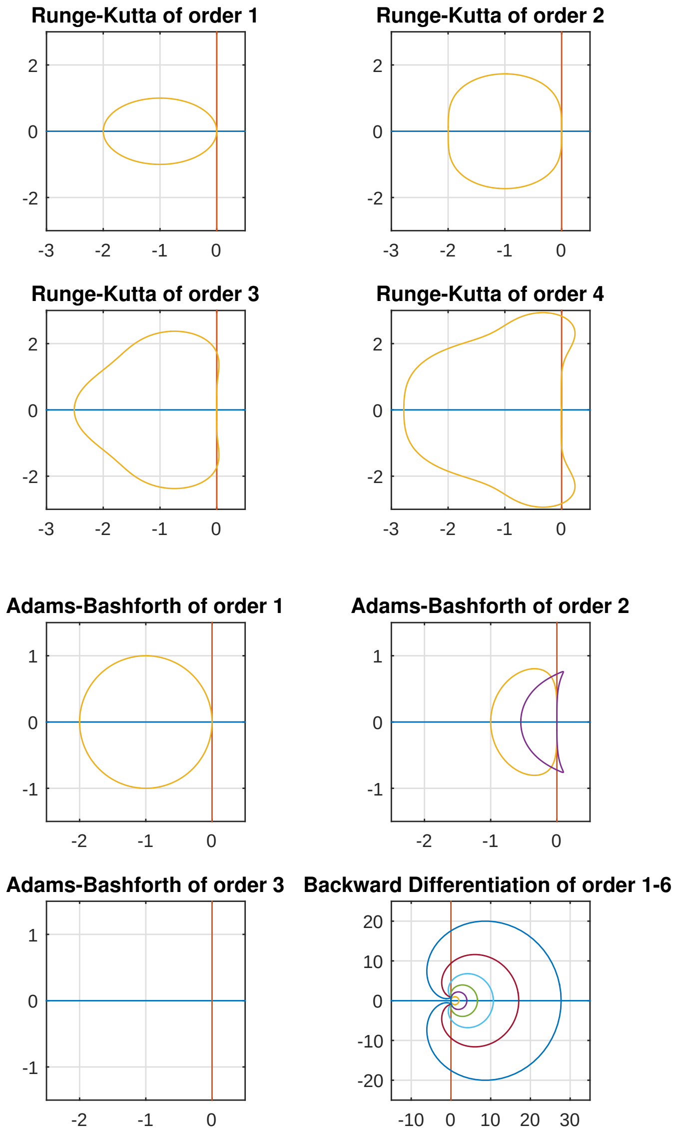

FIGURE 1. Four families of finite difference formulas for ODE have stability regions. The stability zones for backward differentiation are on the outside of the curves, but in the other cases, they are on the inside.

scaled by δt, for test problem of SG model equation below. In Figures 1 below stability regions of eigenvalues for different time integration approaches is displayed.

6. NUMERICAL RESULTS

In this section, we consider SG equation given in (3. 1)-(3. 2) along with initial and boundary conditions for some test problems. The accuracy and efficiency of our presented technique are assessed by using L_∞, L_2, RMS and MAE error norms as defined below.

$$\{ \;\; L_\infty = \|u^{exact} - u^{approx}\|_\infty = \max|u_i^{exact} - u_i^{approx}|, \qquad (6.\ 12)$$

$$\Big\{ \; L_2 = \|u^{exact} - u^{approx}\|_2 = \sqrt{\Delta x \sum_{i=0}^{N} |u_i^{exact} - u_i^{approx}|^2}, \tag{6. 13}$$

$$\Big\{ \; RMS = \sqrt{\frac{1}{N} \sum_{i=1}^{N} |u^{exact} - u^{approx}|^2}, \tag{6. 14}$$

and

$$\Big\{ \; MAE = Max|u^{exact} - u^{approx}|, \tag{6. 15}$$

where $\Delta x = \frac{|b-a|}{N}$.

Results and findings obtained using the suggested RBF-FD approach are compared to past studies. The findings produced are better when compared to other approaches in literature and in good agreement with the exact solution, and hence our approach is very effective and quite efficient measured by accuracy, convergence rate and computing time (CPU Time/Sec), as shown in the following.

6.1. **Test problem 1:** Consider the exact solution of SG equation given in equation (3. 1) as a system of two equations given in equation (3. 2) as under:

$$\begin{aligned} u(x,t) &= 4tan^{-1}[exp\{\gamma(x - Ct) + \beta\}], \\ v(x,t) &= \frac{-4\gamma C[exp\{\gamma(x - Ct) + \beta\}]}{1 + [exp\{\gamma(x - Ct) + \beta\}^2]}. \end{aligned} \tag{6. 16}$$

Where $\gamma = (1 - C^2)^{\frac{-1}{2}}$.

The initial and boundary conditions are extracted from the exact solutions (6. 16). The problem is solved over spatial interval $[-2, 58]$ by the present method using MQ-RBF along with RK-4 time integration scheme, at times $t = 9, 36, 108$, when time step size $\delta t = 0.001$, number of collocation points $N = 121$, number of stencil points $N_x = 25$ and MQ shape parameter $\zeta = 2.4$, while parameters $C = 0.5$ and $\beta = 0$ are used. The L_∞ and L_2, RMSE and MAE error norms at $t = 9, 36, 108$ and CPU Time/Sec are seen in Tables (1-3), also the solitary wave profile in comparison with the exact solution is shown in Figure (2), which confirms the accuracy of RBF-FD meshless method. The results when RBF-FD (MQ) is used, have a good agreement with the exact solution and is better than earlier work in literature [6, 66]. Also the stable and unstable eigenvalue spectrum, that we have calculated for the eigenvalues of the matrix $\mathbf{M}$, scaled by δt related to traditional RK method of order four for this test problem is shown in Figure (3).

TABLE 1. L_∞, L_2, RMS and MAE error norms of u using MQRBF at time $T = 9, 36, 108$ obtained for test problem 1.

Time/Errors	$L_\infty(u)$	$L_2(u)$	$RMS(u)$	$MAE(u)$	C.Time/Sec
9	3.529×10^{-3}	7.045×10^{-3}	2.916×10^{-5}	3.529×10^{-3}	1.0405
36	8.618×10^{-3}	1.680×10^{-2}	7.122×10^{-5}	8.618×10^{-3}	3.9014
108	2.484×10^{-2}	4.614×10^{-2}	2.053×10^{-4}	2.484×10^{-2}	11.2252

TABLE 2. L_∞, L_2, RMS and MAE error norms of v using MQRBF at time $T = 9, 36, 108$ obtained for test problem 1.

Time/Errors	$L_\infty(v)$	$L_2(v)$	$RMS(v)$	$MAE(v)$	C.Time/Sec
9	4.714×10^{-3}	1.008×10^{-2}	3.896×10^{-5}	4.714×10^{-3}	0.9918
36	5.111×10^{-3}	1.148×10^{-2}	4.224×10^{-5}	5.111×10^{-3}	3.8858
108	6.786×10^{-3}	2.011×10^{-2}	5.608×10^{-5}	6.786×10^{-3}	11.1354

TABLE 3. Comparison of L_∞, L_2, RMS and MAE error norms of solution using MQRBF at time $T = 9, 36, 108$ when $\delta t = 0.001$, $N = 121$, $N_x = 25$, $MQ_\zeta = 2.4$, $C = 0.5$ and $\beta = 0$ in $[-2, 58]$ for test problem. 1, given in (6. 16).

Methods	RBF-FD	[66]	[6]	RBF-FD	[66]
Time/Errors	L_∞	L_∞	L_∞	L_2	L_2
9	3.529×10^{-3}	6.836×10^{-3}	4.518×10^{-1}	7.045×10^{-3}	2.683×10^{-2}
36	8.618×10^{-3}	8.032×10^{-2}	$2.293\times10^{+0}$	1.680×10^{-2}	3.113×10^{-1}
108	2.484×10^{-2}	6.253×10^{-1}	$5.120\times10^{+0}$	4.614×10^{-2}	$2.378\times10^{+0}$

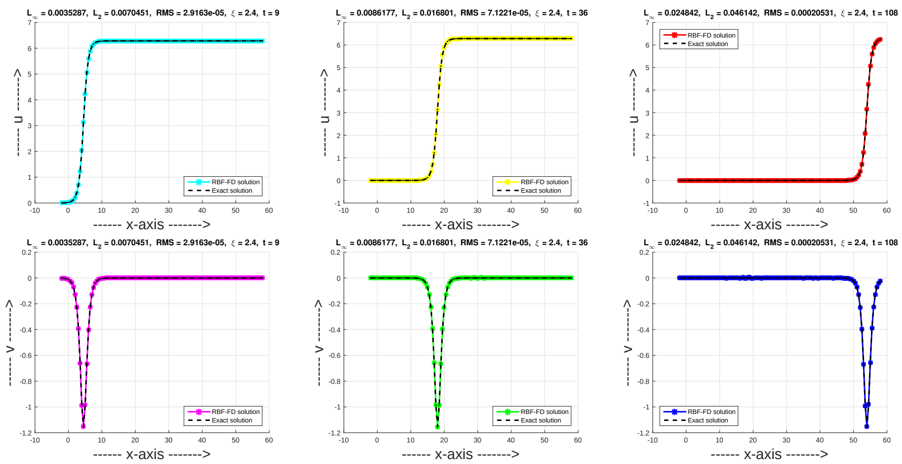

FIGURE 2. Plot of approximate and exact solution of u (up) and v (down) for parameters $C = 0.5$ and $\beta = 0$ at time $T = 9, 36, 108$ with $\delta t = 0.001$ and $N = 121$, $N_x = 25$ using MQ RBF $\zeta = 2.4$ in domain $[-2, 58]$ for test problem. 1, given in (6. 16).

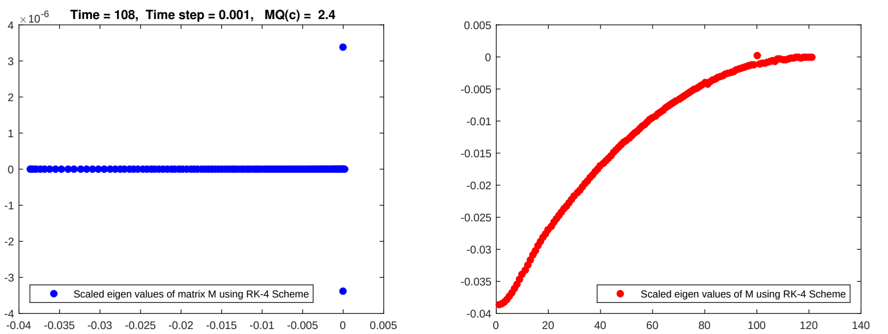

FIGURE 3. Eigenvalues corresponding test problem. 1 related RK-4 formula, they are on the inside the stability zones $(-2.78, 0)$ (left). Eigenvalues corresponding test problem. 1 related RK-4 formula, they are the outside of the stability zones $(-2.78, 0)$, (right).

6.2. **Test problem 2:** Now we consider the exact solution of SG equation given in equation (3. 1) as a system of two equations given in equation (3. 2) as under:

$$u(x,t) = 4tan^{-1}[C^{-1}\cos(\bar{\gamma}Ct)\sec(\bar{\gamma}x)],$$
$$v(x,t) = \frac{-4\bar{\gamma}\cos(\bar{\gamma}Ct)sech(\bar{\gamma}x)}{1+[C^{-1}\sin(\bar{\gamma}Ct)sech(\bar{\gamma}x)]}. \quad (6.\ 17)$$

Where $\bar{\gamma} = (1 - C^2)^{\frac{-1}{2}}$.
The initial and boundary conditions are extracted from the exact solutions (6. 17). The problem is solved over spatial interval $[-10, 10]$ by the present method using MQ-RBF along with RK-4 time integration scheme, at times $t = 1, 10, 20$, when time step size $\delta t = 0.001$, number of collocation points $N = 201$, number of stencil points $N_x = 25$ and MQ shape parameter $\zeta = 2.4$, while parameters $C = 0.5$ and $\beta = 0$ are used. The L_∞ and L_2 error norms at $t = 1, 10, 30$ are seen in Tables (4-6), also the solitary wave profile in comparison with the exact solution is shown in Figure (4), which confirms the accuracy of RBF-FD meshless method. The results when RBF-FD (MQ) is used, have a good agreement with the exact solution and is better than earlier work in literature [6, 66].

TABLE 4. L_∞, L_2, RMS and MAE error norms of u using MQRBF at time $T = 1, 10, 20$ obtained for test problem 2.

Time/Errors	$L_\infty(u)$	$L_2(u)$	$RMS(u)$	$MAE(u)$	C.Time/Sec
1	1.771×10^{-3}	7.605×10^{-3}	8.808×10^{-6}	1.771×10^{-3}	0.1451
10	1.530×10^{-3}	8.859×10^{-3}	7.611×10^{-6}	1.530×10^{-3}	1.1778
20	9.770×10^{-2}	5.737×10^{-1}	4.861×10^{-4}	9.770×10^{-2}	2.346620

TABLE 5. L_∞, L_2, RMS and MAE error norms of v using MQRBF at time $T = 1, 10, 20$ obtained for test problem 2.

Time/Errors	$L_\infty(v)$	$L_2(v)$	$RMS(v)$	$MAE(v)$	C.Time/Sec
1	5.587×10^{-4}	3.179×10^{-3}	2.780×10^{-6}	5.587×10^{-4}	0.1462
10	1.060×10^{-2}	7.297×10^{-2}	5.272×10^{-5}	1.060×10^{-2}	1.2329
20	9.724×10^{-2}	4.361×10^{-1}	4.838×10^{-4}	9.724×10^{-2}	2.4891

TABLE 6. Comparison of L_∞, L_2, RMS and MAE error norms of solution using MQRBF at time $T = 1, 10, 20$ when $\delta t = 0.001$, $N = 201$, $N_x = 25$, $MQ_\zeta = 2.4$, $C = 0.5$ and $\beta = 0$ in $[-10, 10]$ for test problem. 2, given in (6. 17).

Methods	RBF-FD	[66]	[6]	RBF-FD	[66]
Time/Errors	L_∞	L_∞	L_∞	L_2	L_2
1	1.771×10^{-3}	1.474×10^{-3}	0.988×10^{-3}	7.605×10^{-3}	1.252×10^{-2}
10	1.530×10^{-3}	9.215×10^{-3}	0.162×10^{-2}	8.859×10^{-2}	9.402×10^{-2}
20	9.770×10^{-2}	3.038×10^{-1}	0.103×10^{-2}	5.737×10^{-1}	$4.599\times10^{+0}$

6.3. **Test problem 3:** Finally consider the exact solution of SG equation given in equation (3. 1) as a system of two equations given in equation (3. 2) as under:

$$\begin{aligned} u(x,t) &= 4tan^{-1}[Csech(\gamma Ct)\sinh(\gamma x)], \\ v(x,t) &= \frac{-4C^2\gamma sech(\gamma Ct)\tanh(\gamma Ct)\sinh(\gamma x)}{1 + [Csech(\gamma Ct)\sinh(\gamma x)]^2}. \end{aligned} \tag{6. 18}$$

Where $\gamma = (1 - C^2)^{\frac{-1}{2}}$.
The initial and boundary conditions are extracted from the exact solutions (6. 18). The problem is solved over spatial interval $[-20, 20]$ by the present method using MQ-RBF along with RK-4 time integration scheme, at times $T = 2, 10, 20$, when time step size $\delta t = 0.001$, number of collocation points $N = 121$, number of stencil points $N_x = 25$ and MQ shape parameter $\zeta = 2.4$, while parameters $C = 0.5$ and $\beta = 0$ are used. The L_∞ and L_2 error norms at $t = 9, 36, 108$ are seen in Tables (7-9), also the solitary wave profile in comparison with the exact solution is shown in Figure (5), which confirms the accuracy of RBF-FD meshless

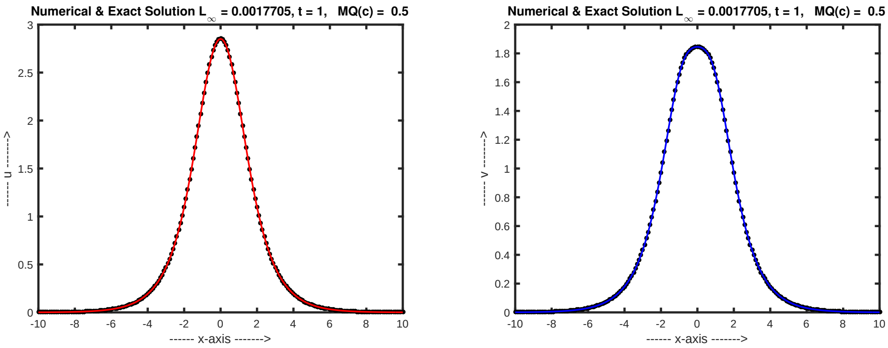

FIGURE 4. Plot of approximate and exact solution of u (left) and v (right) for parameters $C = 0.5$ and $\beta = 0$ at time $T = 1, 10, 20$ with $\delta t = 0.001$ and $N = 201$, $N_x = 25$ using MQ RBF $\zeta = 2.4$ in domain $[-10, 10]$ for test problem 2, given in (6. 17).

method. The results when RBF-FD (MQ) is used, have a good agreement with the exact solution and is better than earlier work in literature [6, 66].

TABLE 7. L_∞, L_2, RMS and MAE error norms of u using MQRBF at time $T = 2, 10, 20$ obtained for test problem 3.

Time/Errors	$L_\infty(u)$	$L_2(u)$	$RMS(u)$	$MAE(u)$	C.Time/Sec
2	2.879×10^{-4}	1.289×10^{-3}	1.432×10^{-6}	2.879×10^{-4}	0.2836
10	3.547×10^{-4}	1.476×10^{-3}	1.764×10^{-6}	3.547×10^{-4}	1.2985
20	2.437×10^{-4}	9.007×10^{-4}	1.213×10^{-6}	2.437×10^{-4}	2.5265

TABLE 8. L_∞, L_2, RMS and MAE error norms of v using MQRBF at time $T = 2, 10, 20$ obtained for test problem 3.

Time/Errors	$L_\infty(v)$	$L_2(v)$	$RMS(v)$	$MAE(v)$	C.Time/Sec
2	7.044×10^{-4}	2.444×10^{-3}	3.504×10^{-6}	7.044×10^{-4}	0.2772
10	8.658×10^{-4}	3.425×10^{-3}	4.307×10^{-6}	8.658×10^{-4}	1.2601
20	7.199×10^{-4}	3.669×10^{-3}	3.582×10^{-6}	7.199×10^{-4}	2.4638

TABLE 9. Comparison of L_∞, L_2, RMS and MAE error norms of solution using MQRBF at time $T = 2, 10, 20$ when $\delta t = 0.001$, $N = 121$, $N_x = 25$, $MQ_\zeta = 2.4$, $C = 0.5$ and $\beta = 0$ in $[-20, 20]$ for test problem. 3, given in (6. 18).

Methods	RBF-FD	[66]	[6]	RBF-FD	[66]
Time/Errors	L_∞	L_∞	L_∞	L_2	L_2
2	2.879×10^{-4}	1.568×10^{-3}	0.127×10^{-3}	1.289×10^{-13}	3.025×10^{-11}
10	3.547×10^{-4}	3.151×10^{-3}	0.191×10^{-3}	1.476×10^{-13}	3.695×10^{-12}
20	2.437×10^{-4}	1.828×10^{-2}	0.251×10^{-3}	9.007×10^{-11}	1.039×10^{-9}

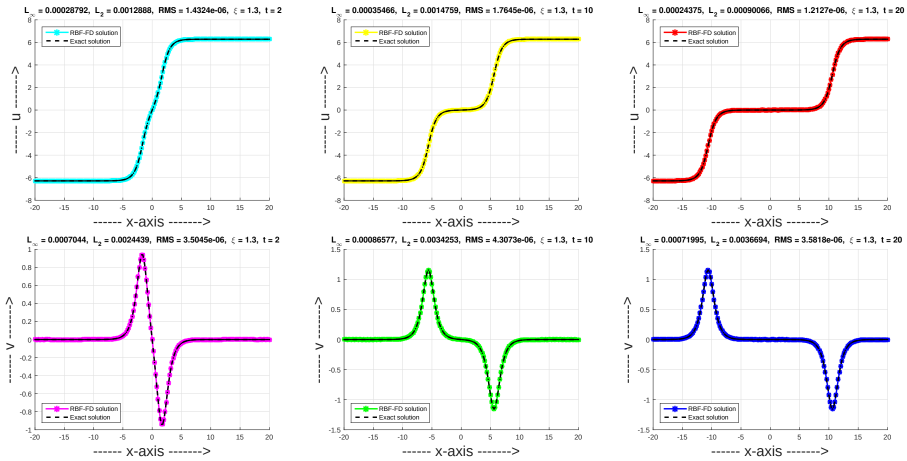

FIGURE 5. Plot of approximate and exact solution of u (up) and v (down) for parameters $C = 0.5$ and $\beta = 0$ at time $T = 2, 10, 20$ with $\delta t = 0.001$ and $N = 121$, $N_x = 25$ using MQ RBF $\zeta = 2.4$ in domain $[-20, 20]$ for test problem. 3, given in (6. 18).

7. CONCLUSION

In this paper radial basis functions for solitary wave model equation like Sine-Gordon (SG) model equation is covered in detail along with some important ideas and definitions. The said model is resolved using a hybrid numerical technique based on radial basis functions (RBFs) and finite differences (FD) called as RBF-FD meshless method. Some non-linear testing problems have been handled. For execution of temporal variable in the given model equation, Runge-Kutta (RK-4) time steeping approach is utilized. We used L_∞, L_2, "RMS" and "MAE" error norms and graphs to assess the accuracy and precision of our proposed method. The advantage of the present method over other existing methods is its local and sparse nature of differentiation matrices. Hence, it is less difficult with low processing cost (CPU time/sec) and more straightforward to treat any higher order nonlinear PDEs. Remarkably, it is clear from our findings that we attain higher accuracy using very small values of the shape parameter ξ than the so large values previously applied in literature.

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10. CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest to report regarding the present study.

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