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On a generalized lcm-sum function

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Abstract. Let $\mathbb{N}$ be the set of all natural numbers. For $r \in \mathbb{N}$, Fogel first considered the greatest r^{th} power common divisor of m and n in $\mathbb{N}$. Denote it by $(m,n)_r$ and call the $r-gcd$ of m and n. Using this notion we introduce the $r-lcm$ of m and n, denoted by $[m,n]_r$. For $s \in \mathbb{R}$, define $L_{s,r}(n)$ to be the sum of $[j,n]_r^s$ for $j = 1,2,3,...,n$. In this paper we obtain an asymptotic formula for the summatory function of $L_{s,r}(n)$. The case $r = 1$ was studied earlier by Alladi, Bordelles and more recently by Ikeda and Matsuoka.

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1. Introduction

Let $\mathbb{N}$ be the set of all natural numbers. For $j, n \in \mathbb{N}$, if $[j,n]$ denotes their least common multiple then Alladi [1] defined the function

(1.1) $$L_s(n) = \sum_{j=1}^{n} [j,n]^s \text{ for } s \in \mathbb{R}$$

and proved;

1.2 Lemma. For $x \geq 1$ and $s \geq 1$,

$$\sum_{n \leq x} L_s(n) = \frac{\zeta(s+2)}{2(s+1)^2\zeta(2)} \cdot x^{2s+2} + \Delta_s(x), \text{ as } x \to \infty$$

where $\zeta(s)$ is the Riemann-zeta function and

(1.3) $\Delta_s(x) = O(x^{2s+1+\varepsilon})$ for any $\varepsilon > 0$.

In [3], it has been proved that for $x > e$ (the value of the exponential function e^x at $x = 1$),

(1.4) $\Delta_1(x) = O\left(x^3 \cdot (\log x)^{\frac{2}{3}} \cdot (\log\log x)^{\frac{4}{3}}\right)$,

which is an improvement of (1.3) in case $s = 1$. Also in the same paper an asymptotic formula for $\sum_{n\leq x} L_{-1}(n)$ is obtained.

Recently Ikeda and Matsuoka [7] have proved the results given below

1.5 Lemma ([7], Theorem 2). If $a \in \mathbb{N}$ and $a \geq 2$, then for $x > e$,

$$\sum_{n\leq x} L_a(n) = \frac{\zeta(a+2)}{2(a+1)^2\zeta(2)} \cdot x^{2a+2} + O(x^{2a+1}(\log x)^{\frac{2}{3}} \cdot (\log\log x)^{\frac{4}{3}}),$$

as $x \to \infty$ in which the implied constant depends on a.

1.6 Lemma ([7], Theorem 3). If $a \in \mathbb{N}$ and $a \geq 2$, then

(1.7) $$\sum_{n=1}^{\infty} L_{-a}(n) = \frac{\zeta(a)}{2}\left(1 + \frac{\zeta^2(a)}{\zeta(2a)}\right),$$

and that for $x \geq 1$,

(1.8) $$\sum_{n\leq x} L_{-a}(n) = \frac{\zeta(a)}{2}\left(1 + \frac{\zeta^2(a)}{\zeta(2a)}\right) - \frac{\zeta(a) \cdot x^{-a+1}\log x}{(a-1)\zeta(a+1)} + O\left(x^{-a+1}\right),$$

as $x \to \infty$, in which the implied constant depends on a.

Observe that Lemma 1.5 improves the order term of Lemma 1.2 in the case $s = a \in \mathbb{N}$ with $a \geq 2$.

In this paper we define a generalized lcm-function, using the notion of the greatest r^{th} power common divisor of $m, n \in \mathbb{N}$, introduced by Fogel in 1900(see. [6], p.134).

Fix $r \in \mathbb{N}$. For $m, n \in \mathbb{N}$, let $(m, n)_r$ denote the *greatest* r^{th} *power common divisor* of m and n, which is called the *r-gcd* of m and n.

Clearly $(m, n)_1 = (m, n)$, the gcd of m and n.

If $m, n \in \mathbb{N}$ are such that $m = \prod_{i=1}^{t} p_i^{\alpha_i}$ and $n = \prod_{i=1}^{t} p_i^{\beta_i}$, where p_i are distinct primes and α_i, β_i are non-negative integers with $\alpha_i + \beta_i > 0$ for $i = 1, 2, 3, ..., t$ then

$(m, n)_r = \prod_{i=1}^{t} p_i^{\gamma_i}$ where $\gamma_i = r \cdot \min\left([\frac{\alpha_i}{r}], [\frac{\beta_i}{r}]\right)$, and $[x]$ denotes the greatest integer not exceeding the real number x.

Define the *generalized least common multiple*, $[m,n]_r$, by
$[m,n]_r = \prod_{i=1}^{t} p_i^{\alpha_i+\beta_i-\gamma_i}$, which we call as the *r-lcm* of m and n.

Note that $[m,n]_1 = [m,n]$, the lcm of m and n and that

(1.9) $$(m,n)_r \cdot [m,n]_r = mn$$

As pointed out by one of the learned referees of this paper the concept of $r - lcm$ of m and n has been mentioned in the article by Z. Bu and Z. Xu [4]. In fact they define $[m,n]_r$ by using (1.9).

Now, the *generalized lcm-sum function*, $L_{s,r}(n)$, is defined by

(1.10) $$L_{s,r}(n) = \sum_{j=1}^{n} [j,n]_r^s \text{ for } s \in \mathbb{R}.$$

One can prove easily, by usual method, that

1.11. Lemma. For $r \in \mathbb{N}$, $s \geq 1$ and $x \geq 1$, we have

$$\sum_{n\leq x} L_{s,r}(n) = \frac{\zeta(rs+2r)}{2(s+1)^2\zeta(2r)} \cdot x^{2s+2} + O(x^{2s+1+\varepsilon}), \text{ as } x \to \infty, \text{ for any } \varepsilon > 0.$$

Since $L_{s,1}(n) = L_s(n)$, defined in (1.1), the case $r = 1$ of Lemma 1.11 gives Lemma 1.2. Also in this case of $r = 1$, if $a \in \mathbb{N}$ with $a \geq 2$, $\Delta_s(x)$ of Lemma 1.2 has been improved for $s = a$ in [3] and the case $s = -a$ is considered in [7], (as given in Lemma 1.5 and Lemma 1.6 above).

Therefore in this paper we consider $r > 1$ and prove the following:

Theorem A. For $a, r \in \mathbb{N}$ with $r > 1$, and $x \geq 1$, we have
$$\sum_{n\leq x} L_{a,r}(n) = \frac{\zeta(ar+2r)}{2(a+1)^2\zeta(2r)} \cdot x^{2a+2} + O_a(x^{2a+1}), \text{ as } x \to \infty,$$
where the implied constant depends on a.

1.12. Remark. It may be noted that Z. Bu and Z. Xu ([5], Theorem 4) have offered a more simple proof of an asymptotic formula for $\sum_{n\leq x} L_{a,r}(n)$, with order term $O(x^{2a+1}\log x)$ by using elementary calculations. But Theorem A, proved here, by a different method(expressing $L_{a,r}(n)$ as a Dirichlet product of two arithmetic functions as given in Lemma 2.14) improves their order term.

Theorem B. For $a, r \in \mathbb{N}$ with $a \geq 2$ and $r > 1$, we have
$$\sum_{n\leq x} L_{-a,r}(n) = \frac{\zeta(ar)}{2}\{1 + F_r(a)\} + O_{a,r}(x^{(-a+1)(2-r)}),$$
where the implied constant depends on a and r; and
$$F_r(a) = \prod_p \left\{ (1 + \frac{2}{p^a} + \frac{3}{p^{2a}} + ... + \frac{2r-1}{p^{(2r-2)a}}) + \frac{2r}{p^{(2r-1)a}(1-p^{-a})} \right\}.$$

2. LEMMAS AND THE PROOF OF THEOREM A.

If $\mu(n)$ is the Mobius function, it well-known that

(2.1) $$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)} \text{ for } s>1;$$

and that

(2.2) $$\sum_{n\leq x}\frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)}+O\left(\frac{1}{x^{s-1}}\right) \text{ for } s>1.$$

Also if $\chi_r(m)$=1 or 0 according as m is the r^{th} power of a positive integer or not, then $\chi_r(m)$ is a multiplicative function and that its Dirichlet series $\sum_{m=1}^{\infty}\frac{\chi_r(m)}{m^s}$ converges absolutely for $s>1$. Further its Euler product representation ([2], Theorem 11.6) is given by

(2.3) $$\sum_{m=1}^{\infty}\frac{\chi_r(m)}{m^s}=\zeta(rs) \text{ for } s>1;$$

and therefore

(2.4) $$\sum_{m\leq x}\frac{\chi_r(m)}{m^s}=\zeta(rs)+O\left(\frac{1}{x^{s-\frac{1}{r}}}\right) \text{ for } s>1.$$

We need the lemma proved in [7].

2.5. Lemma([7], Lemma 4). If $m,a\in\mathbb{N}$ and $S_a(m)=\sum_{k=1}^{m}k^a$, then $S_a(m)=\frac{m^{a+1}}{a+1}+\frac{m^a}{2}+\frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1}B_{k+1}\cdot m^{a-k}$, in which $\{B_k\}_{k=0}^{\infty}$ are Bernoulli numbers defined by $z/(e^z-1)=\sum_{k=0}^{\infty}B_k(z^k/k!)$

2.6. Lemma. For $m,a\in\mathbb{N}$ if $t_{a,r}(m)=\sum_{\substack{k=1\\(k,m)_r=1}}^{m}k^a$, then

(2.7) $$t_{a,r}(m)=m^a\left\{\frac{\phi_r(m)}{a+1}+\frac{1}{2}\varepsilon_r(m)+\frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1}B_{k+1}\cdot\psi_{k,r}(m)\right\},$$

where

(2.8) $$\phi_r(m)=m\cdot\sum_{d^r|m}\frac{\mu(d)}{d^r},$$

(2.9) $$\varepsilon_r(m)=\sum_{d^r|m}\mu(d)$$

and

(2.10) $$\psi_{k,r}(m)=\sum_{d^r|m}\mu(d)\left(\frac{d^r}{m}\right)^k.$$

Proof. First note that $\sum_{d^r|(k,m)} \mu(d) = \begin{cases} 1 & \text{if } (k,m)_r = 1 \\ 0 & \text{if } (k,m)_r > 1 \end{cases}$.

Therefore

$$t_{a,r}(m) = \sum_{0<k\le m} k^a \left(\sum_{d^r|(k,m)} \mu(d) \right) = \sum_{d^r|m} \mu(d) d^{ar} \left(\sum_{0<\delta\le \frac{m}{d^r}} \delta^a \right) = \sum_{d^r|m} \mu(d) d^{ar} S_a\left(\frac{m}{d^r}\right). \tag{2.11}$$

Now using Lemma 2.5 in (2.11) we get

$$\begin{aligned} t_{a,r}(m) &= \sum_{d^r|m} \mu(d) d^{ar} \left\{ \frac{(m/d^r)^{a+1}}{a+1} + \frac{(m/d^r)^a}{2} + \frac{1}{a+1} \sum_{k=1}^{a-1} \binom{a+1}{k+1} B_{k+1} \cdot \left(\frac{m}{d^r}\right)^{a-k} \right\} \\ &= m^a \left\{ \frac{\phi_r(m)}{a+1} + \frac{1}{2}\varepsilon_r(m) + \frac{1}{a+1} \sum_{k=1}^{a-1} \binom{a+1}{k+1} B_{k+1} \cdot \psi_{k,r}(m) \right\}, \end{aligned}$$

proving the lemma. □

2.12. Remark. Observe that $\phi_1(m) = \phi(m)$, the Euler totient function and that $\varepsilon_1(m) = 1$ or 0 according as $m = 1$ or $m > 1$.

2.13. Lemma. If $T_{a,r}(m) = \sum_{j=1}^{m} \frac{j^a}{(j,m)_r^a}$ then $T_{a,r}(m) = \sum_{d^r|m} t_{a,r}\left(\frac{m}{d^r}\right)$.

Proof. If $(j,m)_r = d^r$ then, by definition,

$$\begin{aligned} T_{a,r}(m) &= \sum_{\substack{0<j\le m,\\ (j,m)_r = d^r}} \frac{j^a}{d^{ar}} = \sum_{\substack{0 < d^r\delta \le m,\\ d^r|m \\ (\delta, \frac{m}{d^r})_r = 1}} \frac{(d^r\delta)^a}{d^{ar}} \\ &= \sum_{d^r|m} \sum_{\substack{0<\delta\le \frac{m}{d^r},\\ (\delta,\frac{m}{d^r})_r = 1}} \delta^a = \sum_{d^r|m} t_{a,r}\left(\frac{m}{d^r}\right), \end{aligned}$$

proving the lemma. □

2.14. Lemma. For $a, m \in \mathbb{N}$

$$L_{a,r}(m) = \sum_{\delta|m} \chi_r(\delta) \delta^a M_{a,r}\left(\frac{m}{\delta}\right),$$

where $M_{a,r}(m) = m^a \cdot t_{a,r}(m)$.

Proof. In view of (1.9), Lemma 2.13 and (2.7)

$$\begin{aligned} L_{a,r}(m) &= \sum_{j=1}^{m} \left(\frac{jm}{(j,m)_r} \right)^a = m^a \cdot T_{a,r}(m) \\ &= m^a \sum_{d^r | m} t_{a,r} \left(\frac{m}{d^r} \right) \\ &= \sum_{\delta | m} \chi_r(\delta) \delta^a M_{a,r} \left(\frac{m}{\delta} \right), \end{aligned}$$

proving the lemma. □

To prove Theorem A we have to estimate certain sums involving the functions given in (2.8), (2.9) and (2.10).

2.15. Lemma. If $r > 1$ and $x \geq 1$ then for any $\alpha > 0$

$$\sum_{m \leq x} m^\alpha \phi_r(m) = \frac{x^{\alpha+2}}{(\alpha+2)\zeta(2r)} + O(x^{\alpha+1})$$

Proof. By (2.8) and (2.2), it follows that

$$\begin{aligned} \sum_{m \leq x} \phi_r(m) &= \sum_{d \leq x^{1/r}} \mu(d) \left(\sum_{\delta \leq \frac{x}{d^r}} \delta \right) \\ &= \sum_{d \leq x^{1/r}} \mu(d) \left\{ \frac{(x/d^r)^2}{2} + O\left(\frac{x}{d^r} \right) \right\} \\ &= \frac{x^2}{2} \sum_{d \leq x^{1/r}} \frac{\mu(d)}{d^{2r}} + O\left(x \sum_{d \leq x^{1/r}} \frac{|\mu(d)|}{d^r} \right) \\ &= \frac{x^2}{2\zeta(2r)} + O(x^{1/r}) + O(x) = \frac{x^2}{2\zeta(2r)} + O(x). \end{aligned}$$

Using this formula and the Abel's identity ([2], Theorem 4.2) we get the lemma. □

2.16. Lemma. If $r > 1$ then

(i) $\sum_{m \leq x} m^\alpha \psi_{k,r}(m) = O(x^{\alpha+\frac{1}{r}})$ for $\alpha > k$

(ii) $\sum_{m \leq x} m^\alpha \varepsilon_r(m) = O(x^{\alpha+1})$ for $\alpha > 0$

Proof. By (2.10),

$$(i)\qquad \sum_{m\le x} m^\alpha \psi_{k,r}(m) = \sum_{d^r\delta\le x} d^{r\alpha}\delta^\alpha \frac{\mu(d)}{\delta^k} = \sum_{d\le x^{1/r}} \mu(d)d^{r\alpha}\left(\sum_{\delta\le \frac{x}{d^r}}\delta^{\alpha-k}\right)$$

$$= O\left(\sum_{d\le x^{1/r}} |\mu(d)|d^{r\alpha}\left(\frac{x}{d^r}\right)^{\alpha-k+1}\right)$$

$$= O\left(x^{\alpha-k+1}\sum_{d\le x^{1/r}}\frac{|\mu(d)|}{d^{r(1-k)}}\right)$$

$$= O\left(x^{\alpha-k+1}x^{-(1-k)+\frac{1}{r}}\right) = O\left(x^{\alpha+\frac{1}{r}}\right),$$

$$(ii)\qquad \sum_{m\le x} m^\alpha \varepsilon_r(m) = \sum_{d^r\delta\le x} d^{r\alpha}\delta^\alpha\mu(d) = \sum_{d\le x^{1/r}}\mu(d)d^{r\alpha}\left(\sum_{\delta\le\frac{x}{d^r}}\delta^\alpha\right)$$

$$= O\left(\sum_{d\le x^{1/r}}|\mu(d)|d^{r\alpha}\frac{x^{\alpha+1}}{d^{r\alpha+r}}\right) = O\left(x^{\alpha+1}\sum_{d\le x^{1/r}}\frac{|\mu(d)|}{d^r}\right)$$

$$= O\left(x^{\alpha+1}\right)$$

□

2.17. Lemma. If $r > 1$ and $a \in \mathbb{N}$,

$$\sum_{m\le x} M_{a,r}(m) = \frac{x^{2a+2}}{2(a+1)^2\zeta(2r)} + O_a(x^{2a+1}),$$

where the implied constant depends on a

Proof. Using Lemma 2.15 and Lemma 2.16, we get

$$\sum_{m\le x} M_{a,r}(m) = \sum_{m\le x} m^{2a}\left\{\frac{\phi_r(m)}{a+1} + \frac{1}{2}\varepsilon_r(m) + \frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1}B_{k+1}.\psi_{k,r}(m)\right\}$$

$$= \frac{1}{a+1}\left\{\frac{x^{2a+2}}{(2a+2)\zeta(2r)} + O_a(x^{2a+1})\right\} + O(x^{2a+1})$$

$$+ O\left(\frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1}B_{k+1}x^{2a+\frac{1}{r}}\right)$$

$$= \frac{x^{2a+2}}{2(a+1)^2\zeta(2r)} + O_a(x^{2a+1}) + O\left(x^{2a+\frac{1}{r}}\right),$$

proving the lemma since $r > 1$. Here the implied constant depends on a. □

Proof of Theorem A.
In view of Lemma 2.14 , Lemma 2.17 and (2.4)

$$\begin{aligned}
\sum_{m\le x} L_{a,r}(m) &= \sum_{d\delta\le x}\chi_r(d)d^a M_{a,r}(\delta) = \sum_{d\le x}\chi_r(d)d^a\left(\sum_{\delta\le\frac{x}{d}} M_{a,r}(\delta)\right)\\
&= \sum_{d\le x}\chi_r(d)d^a\left\{\frac{(x/d)^{2a+2}}{2(a+1)^2\zeta(2r)} + O_a\left(\left(\frac{x}{d}\right)^{2a+1}\right)\right\}\\
&= \frac{x^{2a+2}}{2(a+1)^2\zeta(2r)}\sum_{d\le x}\frac{\chi_r(d)}{d^{a+2}} + O_a\left(x^{2a+1}\sum_{d\le x}\frac{\chi_r(d)}{d^{a+1}}\right)\\
&= \frac{x^{2a+2}}{2(a+1)^2\zeta(2r)}\left\{\zeta((a+2)r) + O\left(\frac{1}{x^{a+2-\frac{1}{r}}}\right)\right\} + O_a(x^{2a+1})\\
&= \frac{x^{2a+2}\zeta((a+2)r)}{2(a+1)^2\zeta(2r)} + O\left(x^{a+\frac{1}{r}}\right) + O_a(x^{2a+1}),
\end{aligned}$$

proving the theorem.

3. LEMMAS AND THE PROOF OF THEOREM B.

It is well known that a divisor d of $m \in \mathbb{N}$ is called a *unitary divisor* if $(d, \frac{m}{d}) = 1$. Generalizing this notion, D. Suryanarayana [8] has defined $r - ary\ divisors$ d of $m \in \mathbb{N}$, as those for which $(d, \frac{m}{d})_r = 1$. Denoting the number of r- ary divisors of m by $\tau_r^*(m)$; it has been proved that $\tau_r^*(m)$ is a multiplicative arithmetic function and that

(3.1) $$\tau_r^*(m) = \left(\sum_{d\delta = m, (d,\delta)_r = 1} 1\right) \text{ is such that}$$

$\tau_r^*(p^\alpha) = \alpha + 1$ or $2r$ according as $\alpha < 2r$ or $\alpha \ge 2r$ for any prime p.

Clearly the Dirichlet series $\sum_{m=1}^{\infty}\frac{\tau_r^*(m)}{m^s} = F_r(s)$ converges absolutey for $s > 1$ and therefore has Euler product representation ([2], Theorem 11.6). In view of (3.1), $F_r(s)$ is given by

(3.2) $$F_r(s) = \prod_p\left\{\left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + ... + \frac{(2r-1)}{p^{(2r-2)s}}\right) + \frac{2r}{p^{(2r-1)s}(1-\frac{1}{p^s})}\right\}.$$

We observe that

(3.3) $$F_1(s) = \prod_p\left\{1 + \frac{2}{p^s(1-\frac{1}{p^s})}\right\} = \frac{\zeta^2(s)}{\zeta(2s)}.$$

3.4. Lemma. For a, r and $n \in \mathbb{N}$ with $a \geq 2$,

$\sum_{n=1}^{\infty} L_{-a,r}(n) = \frac{\zeta(ar)}{2}\{1 + F_r(a)\}$, where $F_r(a)$ in as given in (3.2).

Proof. In view of (1.9),

$$\textbf{(3.5)} \qquad L_{-a,r}(n) = \sum_{j=1}^{n}\left(\frac{(j,n)_r}{jn}\right)^a = \frac{1}{n^a}\sum_{\substack{0<j\leq n\\(j,n)_r=d^r}}\frac{d^{ar}}{j^a} = \frac{1}{n^a}\sum_{d^r|n}\sum_{\substack{0<\delta\leq\frac{n}{d^r}\\(\delta,\frac{n}{d^r})_r=1}}\frac{1}{\delta^a}$$

so that

$$\begin{aligned}\textbf{(3.6)} \qquad \sum_{n=1}^{\infty} L_{-a,r}(n) &= \sum_{n=1}^{\infty}\frac{1}{n^a}\left\{\sum_{d^r u=n}\left(\sum_{0<\delta\leq u,(\delta,u)_r=1}\frac{1}{\delta^a}\right)\right\}\\
&= \sum_{d=1}^{\infty}\sum_{u=1}^{\infty}\frac{1}{d^{ar}u^a}\left(\sum_{0<\delta\leq u,(\delta,u)_r=1}\frac{1}{\delta^a}\right)\\
&= \sum_{d=1}^{\infty}\frac{1}{d^{ar}}\left(\sum_{u=1}^{\infty}\frac{1}{u^a}\left\{\sum_{0<\delta\leq u,(\delta,u)_r=1}\frac{1}{\delta^a}\right\}\right)\\
&= \zeta(ar)\sum_{m=1}^{\infty}\frac{1}{m^a}\left(\sum_{u\delta=m,0<\delta\leq u,(\delta,u)_r=1}1\right).\end{aligned}$$

But, by (3.1) and (3.2), we have

$$\begin{aligned}\textbf{(3.7)} \qquad \sum_{m=1}^{\infty}\frac{1}{m^a}\sum_{\substack{u\delta=m\\0<\delta\leq u\\(\delta,u)_r=1}}1 &= 1 + \frac{1}{2}\sum_{m=2}^{\infty}\frac{1}{m^a}\left(\sum_{u\delta=m,(\delta,u)_r=1}1\right)\\
&= 1 + \frac{1}{2}\sum_{m=2}^{\infty}\frac{\tau_r^*(m)}{m^a}\\
&= 1 + \frac{1}{2}\left\{\sum_{m=1}^{\infty}\frac{\tau_r^*(m)}{m^a} - 1\right\}\end{aligned}$$

$$= 1 + \frac{1}{2}\{F_r(a) - 1\}$$
$$= \frac{1}{2}(1 + F_r(a)).$$

Now (3.7) and (3.6) imply Lemma 3.4. □

Proof of Theorem B.
Since

(3.8) $$\sum_{n\le x} L_{-a,r}(n) = \sum_{n=1}^{\infty} L_{-a,r}(n) - S(x), \text{ where } S(x) = \sum_{n>x} L_{-a,r}(n),$$

we estimate $S(x)$.
By (3.5), we have

(3.9)
$$\begin{aligned}
S(x) &= \sum_{n>x}\frac{1}{n^a}\left(\sum_{d^r u=n}\left\{\sum_{0<\delta\le u,(\delta,u)_r=1}\frac{1}{\delta^a}\right\}\right)\\
&= \sum_{d=1}^{\infty}\frac{1}{d^{ar}}\left\{\sum_{u>\frac{x}{d^r}}\frac{1}{u^a}\left(\sum_{0<\delta\le u,(\delta,u)_r=1}\frac{1}{\delta^a}\right)\right\}\\
&= \sum_{d=1}^{\infty}\frac{1}{d^{ar}}\left\{\sum_{u>\frac{x}{d^r}}\frac{1}{u^a}\left(\sum_{0<\delta\le u}\frac{1}{\delta^a}\left(\sum_{t^r|(\delta,u)}\mu(t)\right)\right)\right\}\\
&= \sum_{d=1}^{\infty}\frac{1}{d^{ar}}\left\{\sum_{u>\frac{x}{d^r}}\frac{1}{u^a}\left(\sum_{0<t^r s\le u,t^r|u}\mu(t)\frac{1}{t^{ar}s^a}\right)\right\}\\
&= \sum_{d=1}^{\infty}\frac{1}{d^{ar}}\left\{\sum_{u>\frac{x}{d^r}}\frac{1}{u^a}\left(\sum_{t^r|u}\frac{\mu(t)}{t^{ar}}\left\{\sum_{s\le\frac{u}{t^r}}\frac{1}{s^a}\right\}\right)\right\}\\
&= \sum_{d=1}^{\infty}\frac{1}{d^{ar}}\left\{\sum_{v>\frac{x}{(dt)^r}}\frac{\mu(t)}{t^{2ar}v^a}\left(\sum_{s\le v}\frac{1}{s^a}\right)\right\}\\
&= \sum_{q=1}^{\infty}\frac{1}{q^{2ar}}\left\{\sum_{dt=q}\mu(t)d^{ar}\left(\sum_{v>\frac{x}{q^r}}\frac{1}{v^a}\left\{\sum_{s\le v}\frac{1}{s^a}\right\}\right)\right\}\\
&= \sum_{q=1}^{\infty}\frac{1}{q^{2ar}}\left\{\sum_{dt=q}\mu(t)d^{ar}\left(\sum_{v>\frac{x}{q^r}}\frac{1}{v^a}\left\{\zeta(a) - \frac{v^{1-a}}{a-1} + O(v^{-a})\right\}\right)\right\}\\
&= S_1(x) + S_2(x),
\end{aligned}$$

where $S_1(x)$ and $S_2(x)$ are the sums extended over those $q < x$ and $q \ge x$ respectively.

Now

$$S_1(x)=\sum_{q<x}\frac{1}{q^{2ar}}\left\{\sum_{dt=q}\mu(t)d^{ar}\zeta(a)\frac{(x/q^r)^{-a+1}}{a-1}+O\left(\frac{x}{q^r}\right)^{-a}\right.$$
$$\left.-\frac{(x/q^r)^{-2a+2}}{(2a-2)(a-1)}+O\left(\frac{x}{q^r}\right)^{-2a+1}\right\}$$
$$=\frac{x^{-a+1}\zeta(a)}{a-1}\sum_{q<x}q^{-ar-r}\left(\sum_{dt=q}\mu(t)d^{ar}\right)+O\left(x^{-a}\sum_{q<x}q^{-ar}\left(\sum_{dt=q}\mu(t)d^{ar}\right)\right)$$
$$-\frac{x^{-2a+2}}{(2a-2)(a-1)}\sum_{q<x}q^{-2r}\left(\sum_{dt=q}\mu(t)d^{ar}\right)$$
$$+O\left(x^{-2a+1}\sum_{q<x}q^{-r}\left(\sum_{dt=q}\mu(t)d^{ar}\right)\right)\tag{3.10}$$

But for any $\beta>1$, we find, in view of (2.2), that

$$\sum_{q<x}q^{-\beta}\sum_{dt=q}\mu(t)d^{ar}=\sum_{d<x}d^{ar-\beta}\left(\sum_{t<\frac{x}{d}}\frac{\mu(t)}{t^\beta}\right)=\sum_{d<x}d^{ar-\beta}\left\{\frac{1}{\zeta(\beta)}+O\left(\frac{d}{x}\right)^{\beta-1}\right\}$$
$$=\frac{1}{\zeta(\beta)}\sum_{d<x}d^{ar-\beta}+O\left(x^{1-\beta}\sum_{d<x}d^{ar-1}\right)$$
$$=O_\beta\left(x^{ar-\beta+1}\right)+O\left(x^{ar-\beta+1}\right)=O_\beta\left(x^{ar-\beta+1}\right),\tag{3.11}$$

where the implied constant depends on β.
Taking $\beta=ar+r,\ ar,\ 2r$ and r in (3.11) and using them in (3.10) we get

$$S_1(x)=O_{a,r}\left(x^{-a-r+2}\right)+O_{a,r}\left(x^{-a+1}\right)+O_{a,r}\left(x^{-2a-2r+3+ar}\right)$$
$$+O_{a,r}\left(x^{-2a+ar-r+2}\right)$$
$$=O_{a,r}\left(x^{-2a+ar-r+2}\right)\tag{3.12}$$

Also

$$S_2(x)=O\left(\sum_{q\geq x}\frac{1}{q^{2ar}}\left\{\sum_{dt=q}|\mu(t)|d^{ar}\left(\sum_{v=1}^{\infty}\frac{1}{v^a}\left\{\zeta(a)-\frac{v^{1-a}}{a-1}+O(v^{-a})\right\}\right)\right\}\right)$$
$$=O\left(\sum_{q\geq x}q^{-ar}\left(\sum_{t|q}\frac{|\mu(t)|}{t^{ar}}\right)\right)=O\left(x^{-ar+1}\right)\tag{3.13}$$

By (3.12) and (3.13) we get

$$S(x)=O_{a,r}\left(x^{-2a+ar-r+2}\right)+O\left(x^{-ar+1}\right)=O_{a,r}\left(x^{-2a+ar-r+2}\right)$$
$$=O_{a,r}\left(x^{(-a+1)(2-r)}\right).\tag{3.14}$$

Now Theorem B follows in view of Lemma 3.4, (3.8) and (3.14).

3.15. Remark. Although the main theorems are valid for $r > 1$ only, Lemmas 2.6, 2.13, 2.14 and Lemma 3.4 hold for $r \geq 1$. In fact, taking $r = 1$ in Lemma 3.4, we get (1.7) in view of (3.3).

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