Punjab University Journal of Mathematics (2022),54(3),213-220
https://doi.org/10.52280/pujm.2022.540305

Extended Riemann Integral Equations Involving Generalized $k-$hypergeometric Functions

Aneela Nadir
Department of Mathematics,
NCBA&E, Gulberg III, Lahore, Pakistan,
Email: aneelanadir@yahoo.com

Received: 29 December, 2020 / Accepted: 23 October, 2021 / Published online: 15 March, 2022

Abstract.: This research proves the existence of the solution for the Fredholm integral equation of the first kind. Initially, $k-$Riemann integral equation is considered involving the $k-$hypergeometric function as kernel. $k-$fractional integration defined by Mubeen and Habibullah [16] is used to investigate the solution of the integral equation

$$\int_0^x \frac{(x-t)^{\frac{c}{k}-1}}{\Gamma_k(c)} {}_{q+1}F_{q,k}\left(\begin{matrix}(a_i,k),(b,k)\\(c_i,k)\end{matrix};1-\frac{x}{t}\right) f(t)dt = g(x)$$

where $\lambda, a_i, b, c_i > 0, i = 1, \ldots, q$ and $f \in C_\circ$.
To prove the existence of solution, necessary and sufficient conditions are defined.
Keywords: $k-$Pochhammer symbol, $k-$hypergeometric function, $k-$Fractional Integration, $k-$Riemann integral equation, Fredholm integral equation.

2010 Mathematics Subject Classification. Primary 26A33, 33C45; Secondary 33C60

1. Introduction

Many times, during the solution of complex problems of PDEs in engineering and sciences, we treat important functions dened by improper integrals and series (or innite products). Those functions are generally called special functions. Special functions contain a very old branch of mathematics. For example, trigonometric functions have been studied for over a thousand years, due mainly to their numerous applications in astronomy. Various special functions like Bessel and all cylindrical functions; the Gauss, Kummer, conuent and generalized hypergeometric functions; the classical orthogonal polynomials, the incomplete Gamma and Beta functions, and error functions, the Airy, Whittaker functions, and so on, will provide solutions to integer-order dierential equations and systems, used as mathematical models. However, recently there has been an increasing interest in and widely extended use of dierential equations and systems of fractional order (that is, of

arbitrary order), as better models of phenomena of various physics, engineering, automatization, biology and biomedicine, chemistry, earth science, economics, nature, and so on. The extensions of a number of well-known special functions were investigated recently by several authors $[1-7]$ and $[18-22]$
In this paper, $k-$Riemann integral equation is considered in which the $k-$hypergeometric function is involved as kernel. $k-$fractional integration is used to investigate the solution of the integral equation

$$\int_0^x \frac{(x-t)^{\frac{c}{k}-1}}{\Gamma_k(c)} {}_{q+1}F_{q,k}\left(\begin{matrix}(a_i,k),(b,k)\\(c_i,k)\end{matrix};1-\frac{x}{t}\right)f(t)dt = g(x) \tag{1. 1}$$

where $\lambda, a_i, b, c_i > 0, i = 1, \ldots, q$ and $f \in C_\circ$.
Diaz and Pariguan [8] defined the integral representation of $k-$gamma function and $k-$beta function, respectively given by

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt, \quad \mathrm{Re}(x) > 0, k > 0 \tag{1. 2}$$

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, \qquad x > 0, y > 0 \tag{1. 3}$$

Mubeen and Habibullah [16] defined a $k-$fractional integration by

$$I_k^\alpha(f)(x) = \frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-t)^{(\alpha/k)-1}f(t)dt \qquad k > 0, 0 < x < t < \infty \tag{1. 4}$$

Habibullah [9] investigated the solution of integral equation in which confluent hypergeometric function is involved as a kernel. The author made use of the fractional integral operators to solve the integral equation. Mubeen [15] has solved the confluent $k-$hypergeometric function, which is reduced to Habibullah [9] by taking $k \to 1$. Habibullah [10] used Erdelyi-Kobers fractional integrals and Stieltjes operators to find the solution of integral equations. Love [13] solved the equations involving hypergeometric functions ${}_2F_1$ by using fractional derivatives.
In Podlubny [24], the fractional derivative of f is given by $I^{-c}f$, which is defined as a continuous function φ such that $I^{-c}\varphi$. It follows that if $a, b > 0$ and $I^{-a-b}f$ exists, then $I^{-a}I^{-b}f$ exists and

$$I^{-a}I^{-b} = I^{-b}I^{-a} = I^{-a-b} \tag{1. 5}$$

$$I^aI^aI^{-a}I^{-a}f = f \tag{1. 6}$$

Habibullah and Choudhary [11] proved that if $f \in C_\circ$, then $I^\alpha f$ exists and belongs to $C_\circ$ where $C_\circ$ be the class of those continuous functions on the interval $(0, b)$, open at 0, where $0 < b < \infty$, which are integrable at 0.

2. Main Results

First, an integral operator T_k^λ involving generalized $k-$hypergeometric function ${}_{q+!}F_{q,k}$ is defined, which is reduced to Batman Integral when $k \to 1$ formulated by Love [13].

$$T_k^\lambda\ {}_{q+1}F_q(z) = \int_0^1 \frac{(1-u)^{\frac{\lambda_1}{k}-1}}{\Gamma_k(\lambda_1)}\frac{u^{\frac{c_1}{k}-1}}{\Gamma_k(c_1)} {}_{q+1}F_{q,k}\left(\begin{matrix}(a_i,k),(b,k);\\(c_i,k);\end{matrix}\ zu\right) \tag{2. 7}$$

$$= \frac{1}{\Gamma_k(c_1+\lambda_1)} {}_3F_{2,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_1+\lambda_1,k),(c_j,k);\end{array} z\right), j=2,\ldots,m \quad (2.8)$$

2.1. **Theorem.** If $\lambda_i, a_i, b, c_i > 0, i = 1, ..., q$ are real numbers, then $T_k^{\lambda_1}, T_k^{\lambda_2}, \ldots, T_k^{\lambda_q}$ are continuous integral operators, and ${}_{q+1}F_{q,k}\left[\left(\begin{array}{c}(a_i,k),(b,k)\\(c_i,k)\end{array}\right); zu\right]$ be a generalized $k-$hypergeometric function then

$$T_k^{\lambda_q}\ldots T_k^{\lambda_2}T_k^{\lambda_1}{}_{q+1}F_{q,k}(z) = \frac{1}{\Gamma_k(c_1+\lambda_1)}\frac{1}{\Gamma_k(c_2+\lambda_2)}\cdots\frac{1}{\Gamma_k(c_q+\lambda_q)}{}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_i+\lambda_i,k);\end{array} z\right) \quad (2.9)$$

Proof Applying integral operator $T_k^{\lambda_1}$ on ${}_{q+1}F_{q,k}(z)$

$$T_k^{\lambda_1}{}_{q+1}F_{q,k}(z) = \int_0^1 \frac{(1-u)^{\frac{\lambda_1}{k}-1}}{\Gamma_k(\lambda_1)}\frac{u^{\frac{c_1}{k}-1}}{\Gamma_k(c_1)}{}_{q+1}F_{q,k}\left[\begin{array}{c}(a_i,k),(b,k)\\(c_i,k)\end{array}; zu\right]du$$
$$= \frac{1}{\Gamma_k(\lambda_1)}\frac{1}{\Gamma_k(c_1)}\int_0^1(1-u)^{\lambda_1/k^{-1}}u^{c_1/k-1}\sum_{n=0}^{\infty}\frac{(a_i)_{n,k}(b)_{n,k}}{(c_i)_{n,k}}\frac{z^n u^n}{n!}du$$
$$= \frac{1}{\Gamma_k(\lambda_1)}\frac{1}{\Gamma_k(c_1)}\int_0^1(1-u)^{\lambda_1/k^{-1}}u^{c_1/k^{+n-1}}\sum_{n=0}^{\infty}\frac{(a_i)_{n,k}(b)_{n,k}}{(c_i)_{n,k}}\frac{z^n}{n!}du$$

By changing the order of integration and summation, and applying $k-$Beta function, following required result is obtained:

$$T_k^{\lambda_1}{}_{q+1}F_{q,k}(z) = \frac{1}{\Gamma_k(c_1+\lambda_1)}{}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_1+\lambda_1,k),(c_j,k);\end{array} z\right)$$

where $i = 1,\ldots,q, j = 2,\ldots,q$

Let $\frac{1}{\Gamma_k(c_1+\lambda_1)}{}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_1+\lambda_1,k),(c_j,k);\end{array} z\right) = F'$

Now, by applying operator on F' and iterating the operators up to q^{th} times, following required result is obtained.

Substitute $z = \frac{t-x}{t}, u = \frac{s-t}{x-t}$ in (2.7) to obtain

$$\frac{1}{\Gamma_k(\lambda_1)}\frac{1}{\Gamma_k(c_1)}\int_t^x(x-s)^{\frac{\lambda_1}{k}-1}(s-t)^{\frac{c_1}{k}-1}{}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_i,k);\end{array} 1-\frac{s}{t}\right)ds$$
$$= \frac{(x-t)^{\frac{\lambda_1}{k}+\frac{c_1}{k}-1}}{\Gamma_k(c_1+\lambda_1)}{}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_1+\lambda_1,k),(c_j,k);\end{array} 1-\frac{x}{t}\right) \quad (2.10)$$

By continuing this way, at the q^{th} step, substitute $z = \frac{t-x}{t}, u = \frac{s-t}{x-t}$

$$\frac{1}{\Gamma_k(\lambda_q)}\frac{1}{\Gamma_k(c_q)}\int_t^x(x-s)^{\frac{\lambda_q}{k}-1}(s-t)^{\frac{c_q}{k}-1}.$$
$${}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_1+\lambda_1,k),(c_2+\lambda_2,k),\ldots,(c_{q-1}+\lambda_{q-1},k),(c_q,k)\,ds;\end{array} 1-\frac{s}{t}\right)ds$$
$$= \frac{(x-t)^{\frac{\lambda_q}{k}+\frac{c_q}{k}-1}}{\Gamma_k(c_q+\lambda_q)}{}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_i+\lambda_i,k);\end{array} 1-\frac{x}{t}\right) \quad (2.11)$$

2.2. **Theorem.** If $\lambda_i, c_i > 0, i = 1, \ldots, q$ are real numbers and $f \in C_\circ$ then the solution to the $k-$Riemann integral equation

$$\int_0^x \frac{(x-t)^{\frac{c}{k}-1}}{\Gamma_k(c)} {}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_i,k);\end{array} 1-\frac{x}{t}\right) f(t)dt = g(x) \qquad (2.\ 12)$$

is $I_k^{\lambda_q} \ldots I_k^{\lambda_2} I_k^{\lambda_1} H_k\left(a_i, b, c_i\right) f(x) = H_k\left(a_i, b, c_i+\lambda_i\right) f(x)$

Proof: Consider $H_k(a_i, b, c_i) f(x)$

$$= \int_0^x \frac{(x-t)^{\frac{c_1}{k}-1}}{\Gamma_k\left(c_1\right)} {}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_i,k);\end{array} 1-\frac{x}{t}\right) f(t)dt \qquad (2.\ 13)$$

Now, by applying $I_k^{\lambda_1}$ on both sides, this implies that

$$I_k^{\lambda_1} H_k(a_i, b, c_i) f(x) = \int_0^x \frac{(x-s)^{\frac{\lambda_1}{k}-1}}{\Gamma_k\left(\lambda_1\right)} ds \int_0^s \frac{(s-t)^{\frac{c_1}{k}-1}}{\Gamma_k\left(c_1\right)} {}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_i,k);\end{array} 1-\frac{s}{t}\right) f(t)dt \qquad (2.\ 14)$$

Changing the order of integration and substituting the result from (2.10), we get

$$I_k^{\lambda_1} H_k(a_i, b, c_i) f(x) = \int_0^x \frac{(x-t)^{\frac{\lambda_1}{k}+\frac{c_1}{k}-1}}{\Gamma_k\left(c_1+\lambda_1\right)} \cdot$$
$${}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_1+\lambda_1,k),(c_2,k),\ldots,(c_q,k);\end{array} 1-\frac{x}{t}\right) f(t)dt \qquad (2.\ 15)$$

$$I_k^{\lambda_1} H_k(a_i, b, c_i) f(x) = H_k\left(a_i, b, c_1+\lambda_1, c_2, \ldots, c_q\right) f(x) \qquad (2.\ 16)$$

Continuing this way and $I_k^{\lambda_q}$ applying on (2.16) and changing the order of integration by using Fubinis theorem [10] and using the result (2.11), we get

$$I_k^{\lambda_q} H_k(a_i, b, c_1+\lambda_1, \ldots, c_{q-1}+\lambda_{q-1}, c_q) f(x)$$
$$= \int_0^x \frac{(x-t)^{\frac{\lambda_q}{k}+\frac{c_q}{k}-1}}{\Gamma_k\left(c_q+\lambda_q\right)} {}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_i+\lambda_i,k);\end{array} 1-\frac{x}{t}\right) f(t)dt \qquad (2.\ 17)$$

$$I_k^{\lambda_q} H_k(a_i, b, c_1+\lambda_1, \ldots, c_{q-1}+\lambda_{q-1}, c_q) f(x) = H_k\left(a_i, b, c_i+\lambda_i\right) f(x) \qquad (2.\ 18)$$

Combining all the results, it implies that

$$I_k^{\lambda_q} \ldots I_k^{\lambda_2} I_k^{\lambda_1} H_k\left(a_i, b, c_i\right) f(x) = H_k\left(a_i, b, c_i+\lambda_i\right) f(x) \qquad (2.\ 19)$$

3. Formal Solution:

To consider the $k-$hypergeometric integral (1.1), let us formulate an integral equation with a given function g and the function f to be determined.

$$H_k\left(a_i, b, c_i\right) f(x) = \int_0^x \frac{(x-t)^{\frac{c}{k}-1}}{\Gamma_k(c)} {}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(c_i,k);\end{array} 1-\frac{x}{t}\right) f(t)dt = g(x) \qquad (3.\ 20)$$

Where $a_i, c_i > 0, i = 1, \ldots, q$. Applying formally the fractional integral $I_k^{a_1}$ on (3.20), we get

$$I_k^{a_1} H_k\left(a_i, b, c_i\right) f(x) = I_k^{a_1} g(x) \tag{3. 21}$$

Using (2.17), we find that

$$H_k\left(a_i, b, c_1 + a_1, c_2, \ldots, c_q\right) f(x) = I_k^{a_1} g(x) \tag{3. 22}$$

Applying then the fractional integral operator $I_k^{a_2}$ on (3.22)

$$I_k^{a_2} H_k\left(a_i, b, c_1 + a_1, c_2, \ldots, c_q\right) f(x) = I_k^{a_2} I_k^{a_1} g(x) \tag{3. 23}$$

Using (2.17), we obtain

$$H_k\left(a_i, b, c_1 + a_1, c_2 + a_2, c_3, \ldots, c_q\right) f(x) = I_k^{a_2} I_k^{a_1} g(x) \tag{3. 24}$$

By continuing this way and applying the fractional integral operator

$$I_k^{a_q} H_k\left(a_i, b, c_1 + a_1, \ldots, c_{q-1} + a_{q-1}, c_q\right) f(x) = I_k^{a_q} \ldots I_k^{a_2} I_k^{a_1} g(x)$$

This leads to the following equations:

$$H_k\left(a_i, b, c_1 + a_1, \ldots, c_q + a_q\right) f(x) = I_k^{a_q} \ldots I_k^{a_2} I_k^{a_1} g(x)$$

$$I_k^{c_q} H_k\left(a_i, b, c_1 + a_1, c_2 + a_2, \ldots, c_{q-1} + a_{q-1}, a_q\right) f(x) = I_k^{a_q} \ldots I_k^{a_2} I_k^{a_1} g(x) \tag{3. 25}$$

$$I_k^{c_q} I_k^{c_{q-1}} H_k\left(a_i, b, c_1 + a_1, \ldots, c_{q-2} + a_{q-2}, a_{q-1}, a_q\right) f(x) = I_k^{a_q} \ldots I_k^{a_2} I_k^{a_1} g(x) \tag{3. 26}$$

By continuing this way and iterating q times, we get

$$I_k^{c_q} I_k^{c_{q-1}} \ldots I_k^{c_1} H_k\left(a_i, b, a_i\right) f(x) = I_k^{a_q} \ldots I_k^{a_2} I_k^{a_1} g(x)$$

Applying the fractional integral operator $I_k^{-c_q}$ on (3.25) and using the fact $I^{\alpha} I^{-\alpha} f = I^{-\alpha} I^{\alpha} f = f$, we obtain

$$I_k^{c_{q-1}} \ldots I_k^{c_1} H_k\left(a_i, b, a_i\right) f(x) = I_k^{-c_q} I_k^{a_q} \ldots I_k^{a_2} I_k^{a_1} g(x) \tag{3. 27}$$

Continuing this way, applying the fractional derivative q times, we obtain

$$H_k\left(a_i, b, a_i\right) f(x) = I_k^{-c_1} I_k^{-c_2} \ldots I_k^{-c_q} I_k^{a_q} \ldots I_k^{a_2} I_k^{a_1} g(x) \tag{3. 28}$$

4. Solution of the $k-$Riemann Integral Equation involving ${}_{q+1}F_{q,k}$

We now attempt to find the solution of the $k-$fractional integral equation given in (1.1)

4.1. **Theorem.** If $f \in C_0$, then the solution to the $k-$Riemann integral equation (1.1) is

$$f(x) = x^{-b} I_k^{-a} x^b I_k^{-c_1} \ldots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{-a_q} \ldots I_k^{a_2} I_k^{a_1} g(x) \tag{4. 29}$$

4.2. **Proof.** Consider the equation (3.27). On expanding, we get

$$\int_0^x \frac{(x-t)^{\frac{a}{k}}}{\Gamma_k(a)} {}_{q+1}F_{q,k}\left(\begin{array}{c}(a_i,k),(b,k);\\(a_i,k);\end{array} 1-\frac{x}{t}\right) f(t)dt = I_k^{-c_1}\dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{a_q}\dots I_k^{a_2} I_k^{a_1} g(x) \tag{4. 30}$$

This implies that

$$\int_0^x \frac{(x-t)^{\frac{a}{k}}}{\Gamma_k(a)} {}_{q+1}F_{q,k}\left(\begin{array}{c}(b,k);\\-;\end{array} 1-\frac{x}{t}\right) f(t)dt = I_k^{-c_1}\dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{a_q}\dots I_k^{a_2} I_k^{a_1} g(x) \tag{4. 31}$$

Since $F(-,b,-;z) = (1-z)^{-b}$,we obtain the following result which simplifies to the solution of the equation as shown below

$$\int_0^x \frac{(x-t)^{\frac{a}{k}}}{\Gamma_k(a)} \left(1-\left(1-\frac{x}{t}\right)\right)^{-b} f(t)dt = I_k^{-c_1}\dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{-a_q}\dots I_k^{a_2} I_k^{a_1} g(x) \tag{4. 32}$$

$$\int_0^x \frac{(x-t)^{\frac{a}{k}-1}}{\Gamma_k(a)} \left(\frac{x}{t}\right)^{-b} f(t)dt = I_k^{-c_1}\dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{-a_q}\dots I_k^{a_2} I_k^{a_1} g(x) \tag{4. 33}$$

$$x^{-b}\int_0^x \frac{(x-t)^{\frac{a}{k}-1}}{\Gamma_k(a)} t^b f(t)dt = I_k^{-c_1}\dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{-a_q}\dots I_k^{a_2} I_k^{a_1} g(x) \tag{4. 34}$$

$$x^{-b} I_k^a\left(x^b f(x)\right) = I_k^{-c_1}\dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{-a_q}\dots I_k^{a_2} I_k^{a_1} g(x) \tag{4. 35}$$

$$I_k^a\left(x^b f(x)\right) = x^b I_k^{-c_1}\dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{-a_q}\dots I_k^{a_2} I_k^{a_1} g(x) \tag{4. 36}$$

$$x^b f(x) = I_k^{-a} x^b I_k^{-c_1}\dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{-a_q}\dots I_k^{a_2} I_k^{a_1} g(x) \tag{4. 37}$$

$$f(x) = x^{-b} I_k^{-a} x^b I_k^{-c_1}\dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{-a_q}\dots I_k^{a_2} I_k^{a_1} g(x) \tag{4. 38}$$

5. NECESSARY AND SUFFICIENT CONDITIONS

To prove the existence of solution, consider the solution, which includes $k-$fractional integral operator and functions.

$$f(x) = x^{-b} I_k^{-a} x^b I_k^{-c_1}\dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{-a_q}\dots I_k^{a_2} I_k^{a_1} g(x) \tag{5. 39}$$

Now, we know that $g(x) \in C_0$. So, from [9], we get that

$$I_k^{a_1} g(x) \in C_0 \tag{5. 40}$$

Now if $I_k^{a_1} g(x) = h(x)$ and $h(x) \in C_0$, again from [9], we get

$$I_k^{a_2} h(x) = I_k^{a_2} I_k^{a_1} g(x) \in C_0 \tag{5. 41}$$

Continuing this way since we can say that

$$I_k^{a_q} \dots I_k^{a_2} I_k^{a_1} g(x) \in C_0 \tag{5. 42}$$

Now, since $c_i > 0, i = 1, \dots, q$, it implies that

$$I_k^{-c_1} \dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{-a_q} \dots I_k^{a_2} I_k^{a_1} g(x) \in C_0 \tag{5. 43}$$

Since $b > 0$ so x^b is a continuous function, and we know that product of continuous function is continuous,

$$\Rightarrow x^b I_k^{-c_1} \dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{a_q} \dots I_k^{a_2} I_k^{a_1} g(x) \in C_0 \tag{5. 44}$$

$$\Rightarrow I_k^{-a} x^b I_k^{-c_1} \dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{a_q} \dots I_k^{a_2} I_k^{a_1} g(x) \in C_0, \quad a > 0 \tag{5. 45}$$

$$\Rightarrow x^{-b} I_k^{-a} x^b I_k^{-c_1} \dots I_k^{-c_{q-1}} I_k^{-c_q} I_k^{a_q} \dots I_k^{a_2} I_k^{a_1} g(x) \in C_0 \tag{5. 46}$$

From (5.38), we can say that

$$f \in C_0 \tag{5. 47}$$

so $f(x)$ exists and it is continuous.

6. Conclusion

In the present paper, the solution of integral equation in which $k-$hypergeometric function is involved as a kernel, is investigated. The author made use of the $k-$fractional integral operators defined by Mubeen and Habibullah [16] to solve the integral equation, which is reduced to Habibullah [10] by taking $k \to 1$. On account of the general nature and usefulness of the fractional integral operators involved in the present study, the findings of the paper are believed to be useful in fields of applied Mathematics.

References

[1] P. Agarwal, J. Tariboon and S. K. Ntouyas, *Some generalized Riemann-Liouville $k-$fractional integral inequalities*, Journal of Inequalities and Applications. **2016,** No.1 (2016) 122.

[2] P. Agarwal and J. E. Restrepo, *An extension by means of $\omega-$weighted classes of the generalized Riemann-Liouville $k-$fractional integral inequalities*, Journal of Mathematical Inequalities. **14,** No.1 (2020) 35-46.

[3] B. Ahmad and J. J. Nieto, *Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions*, Boundary Value Problems, **2011,** No.1 (2011) 36.

[4] E. Agyuz and M. Acikgoz, *On the study of modified $(p, q)-$Bernstein polynomials and their applications*, International Journal of Advanced and Applied Sciences, **5**(2018) 61-65.

[5] A. Bhargava1, A. Srivastava and R. Mukherjee, *On Generalized Fractional Integration of Aleph $(\aleph)-$Function*, International Journal of Applied and Computational Mathematics, **3** (2017) 233-241.

[6] A. Bhargava1, A. Srivastava and R. Mukherjee, *On Generalized Fractional Integration of $I-$function*, Caspian Journal of Applied Mathematics, Ecology and Economics, **3,** No. 2 (2015) 51-59.

[7] A. Bhargava, G, Agarwal and S. S. Sharma, *On fractional integral formulas involving Srivastava's polynomials and multivariable $I-$function*, Mathematics in Engineering, Science and Aerospace (MESA) **11,** No.2 (2020)

[8] R. Diaz and E. Pariguan, *On hypergeometric functions and Pochhammer $k-$symbol*, Divulgacione Matematicas **15,** No.2 (2007) 179-192.

[9] G. M. Habibullah, *Some Integral Equations Involving Confluent Hypergeometric Functions*, The Yokohama Mathematival Journal **19** (1971) 35-43.

[10] G. M. Habibullah, *Some integral operators with hypergeometric functions as Kernels*, Bulletin mathmatique de la Socit des Sciences Mathmatiques de la Rpublique Socialiste de Roumanie, **21** (1977) 293-300.

[11] G. M. Habibullah and M. D. Choudhary, *On fractional integration*, Mathematics Forum **3,** No.4 (1993) 39-41.

[12] D. Korkmaz-Duzgun, and E. Erkus-Duman, *Generating functions for $k-$hypergeometric functions.* Int. J. Appl. Phys. Math, **9** (2019)119-126.

[13] E. R. Love, *Some integral equations involving hypergeometric functions*. Proceedings of the Edinburgh Mathematical Society. **15** (1967) 169-198.

[14] M. Mansour, *Determining the generalized $k-$gamma function $\Gamma_k(x)$ by functional equations*, International Journal of Contemporary Mathematical Sciences **4** (2009) 1037-1042.

[15] S. Mubeen, *Solution of some integral equations involving confluent $k-$hypergeometric functions*, Applied Mathematics. **4** (2013) 9-11.

[16] S. Mubeen and G. M. Habibullah, *An integral representation of some $k-$hypergeometric Functions*, International Mathematical Forum. **7**, No.4, (2012) 203-207.

[17] S. Mubeen and G. M. Habibullah, *$k-$fractional integrals and application*, International Journal of Contemporary Mathematical Sciences. **7,** No.2 (2012) 89-94.

[18] A. Nadir, A. Khan and M. Kalim, *Integral transform of the generalized Mittag-Leffler function*, Applied Mathematical Sciences. **8** (2014) 5145-5154.

[19] A. Nadir and A. Khan, *Marichev-Saigo-Maeda differential operator and generalized incomplete hypergeometric functions*, Punjab University Journal of Mathematics. **51,** No.2 (2018)123-129.

[20] A. Nadir and A. Khan, *Caputo MSM fractional Differentiation of Extended Mittag-Leffler function*, International Journal of Advance and Applied Sciences. **5,** No.10 (2018) 28-34.

[21] A. Nadir and A. Khan, *Fractional integral operator associated with extended Mittag-Leer function*, International Journal of Advanced and Applied Sciences. **6,** No.2 (2019)15.

[22] A. Nadir and A. Khan, *Extended Mittag-Leffler Functions Associated with Weyl Fractional Calculus operators*, Punjab University Journal of Mathematics.**52,** No.4 (2020) 53-64.

[23] K. S. Nisar, G. Rahman, J. Choi, S. Mubeen and M. Arshad, *Generalized hypergeometric $k-$functions via $(k,s)-$fractional calculus*. J. Nonlinear Sci. Appl, **10** (2017)1791-1800.

[24] I. Podlubny, *Geometric and Physical Interpretation of Fractional Integration and Fractional Differentiation*, Slovakia. 2008.

[25] G. Rahman, M. Arshad and S. Mubeen, *Some results on a generalized hypergeometric $k-$functions*, Bull. Math. Anal. Appl, **8** (2016) 6-77.

[26] M. Z. Sarikaya, Z. Dahmani, M. E. Kiris and F. Ahmad, *$(k,s)-$Riemann-Liouville fractional integral and applications*. Hacettepe Journal of Mathematics and Statistics. **45,** No.1 (2016) 77-89.

[27] A Srivastava, R. Mukherjee and A. Bhargav, *A Unified Approach to Fractional Calculus Pertaining to $I-$ Functions*, International Journal of Mathematics and Computer Applications Research (IJMCAR). **4,** No.1(2014) 31-42.