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Fuzzy Bipolar Soft Quasi-ideals in Ordered Semigroups

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Abstract. In this paper, we introduce the concept of fuzzy bipolar soft quasi-ideals in ordered semigroup theory. First some characteristics of the structure are examined and hence a few useful results are established. It is proved, among others, that the concepts of fuzzy bipolar soft bi-ideal and fuzzy bipolar soft quasi-ideal in regular ordered semigroups coincide. In addition, fuzzy bipolar soft quasi-ideals over ordered semigroups are linked with the ordinary quasi-ideals. Thereafter, a few classes of ordered semigroups are characterized in terms of their fuzzy bipolar soft left, fuzzy bipolar soft right and fuzzy bipolar soft quasi-ideals, and thus some important characterization theorems are established. We also define fuzzy bipolar soft semiprime quasi-ideals and characterize completely regular ordered semigroups by their fuzzy bipolar soft (semiprime) quasi-ideals. It is proved that an ordered semigroup S is completely regular if and only if every fuzzy bipolar soft quasi-ideal λ_A over S is a fuzzy bipolar soft semiprime quasi-ideal.

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1. Introduction

Whenever the real-world problems (ranging from engineering to medical sciences to social sciences) and their modeling come under discussion, we face the existence of uncertainty in the data. Thus to deal with the data having uncertainty, Zadeh [36] initiated the theory of fuzzy sets. Within few years of the initiation of Zadeh's theory a natural question of possible connection between algebraic structures and Zadeh's theory raised in mind.

Therefore, Rosenfeld [31] connected group theory with fuzzy set theory by introducing fuzzy group theory and, following his footprints, Kuroki [16-21] defined the concept of fuzzy semigroup theory and investigated fuzzy ideals, fuzzy (generalized) bi-ideals, fuzzy interior-ideals, fuzzy semiprime ideals and fuzzy semiprime quasi-ideals. Similarly, the concept of fuzzy quasi-ideals in semigroups was introduced by Ahsan et al. [1]. Later the concepts given in [1, 16-21] were further studied in the structure of ordered semigroups by Kehayopulu and Tsingelis [13-15], Kehayopulu [9], Kehayopulu et al. [12], and Shabir and Khan [32]. Obviously, in the real-world problems many a time an information about a specific phenomenon has two aspects, that is, the presence or absence of a particular property. In other words, the information is bipolar with a positive and a negative pole, where the positive pole represents what is granted to be possible and the negative pole represents what is reflected to be impossible. Therefore, Zadeh's theory of fuzzy sets [36] is inappropriate to deal with such problems. Thus to handle the problems that involve a bipolar data, several generalizations of Zadeh's theory are attempted. One of these generalizations is known as bipolar fuzzy set [37] whose membership degree range is [−1,1]. An element may have 0 membership degree or its membership degree may lie in $(0, 1]$ or in $[-1, 0)$ and, in these cases, we say that the element is irrelevant to the corresponding property or somewhat satisfies the property or somewhat satisfies the implicit counter-property, respectively. The motivation behind this notion was to remove the inadequacy of parameterization tools in the existing theories of fuzzy mathematics. Later, Molodtsov [25] further generalized Zadeh's theory [36] and introduced soft set theory to deal with uncertainties which cannot be handled by the existing mathematical tools. In addition, Maji et al. [23] studied both the concepts of [25] and [36], and came up with the idea of fuzzy soft set theory. Later, Maji et al. [24] applied the concept of [25] to handle decision making problems. Even though rough set theory [22, 29] can deal well with the issues of parameterization, whereas its hybrid structures such as fuzzy rough sets [30] can also be used for the purpose of dealing with the fuzziness of data, but the addition of any further factor such as bipolarity of information makes it complex to be used. Instead, soft set theory has no restrictions in the process of making approximations for an object and, therefore, this theory is more applicable than the other ones. Akram et al. [2] studied uncertainty and vagueness in K-algebra using the concept of bipolar fuzzy soft set theory [35]. Muhiuddin and Mahboob [26] studied int-soft (left, right, interior, bi-) ideals over soft sets in the structure of ordered semigroups. In the same fashion, Muhiuddin et al. [27] introduced the concept of an $(\in, \in \vee(k^*, q_k))$-fuzzy semiprime subset in ordered semigroups and investigated several related properties. Talee et al. [33] applied hesitant fuzzy set theory to ordered Γ-semigroups and studied hesitant fuzzy ideals and their related properties. Likewise, Jun et al. [6] introduced the concept of hesitant fuzzy semigroups with a frontier and examined some properties of the notion.

As mentioned above, Kuroki [21] and Ahsan et al. [1] studied fuzzy quasi-ideals in semigroup theory and investigated the basic properties of semigroups in terms of their fuzzy quasi-ideals. Likewise, Shabir and Khan [32] studied the same ideals in ordered semigroup theory and characterized several classes of ordered semigroups by these ideals. As discussed earlier, there are several generalizations and extensions of Zadeh's fuzzy set theory, for example, interval-valued fuzzy sets, intutionistic fuzzy sets, vague sets, fuzzy soft sets, hesitant fuzzy sets and fuzzy rough sets. It is worth mentioning that all these theories have useful applications in various fields that show the importance of these structures. In 2014, Naz and Shabir [28] presented another extension of fuzzy set theory named fuzzy bipolar soft (FBS) set, which is a blending of fuzzy and bipolar soft set theories, and discussed its application in decision making problems. The concept of [28] was redefined by the authors and applied to ordered semigroup theory. Thus the concept of FBS ideal theory in ordered semigroups was initiated [3-5]. Motivated by the research work of Kuroki, Ahsan et al., and Shabir and Khan, we apply in the article in hand the concept of FBS set theory to ordered semigroups and introduce the notion of FBS quasi-ideals. First a few fundamental characteristics of the notion are examined. In this respect, we prove among others that (i) every FBS quasi-ideal over an ordered semigroup S is an FBS ordered semigroup; (ii) every FBS (left, right) ideal λ_A over S is an FBS quasi-ideal; (iii) an FBS set λ_A over S is an FBS quasi-ideal over S iff its (r,t)-level subset is a quasi-ideal of S; (iv) every FBS quasi-ideal λ_A over S is an FBS bi-ideal; (v) every FBS bi-ideal λ_A over a regular ordered semigroup S is an FBS quasi-ideal; and (vi) a non-empty subset Q of S is a quasi-ideal of S iff the FBS characteristic function χ_A^Q of Q is an FBS quasi-ideal over S. Thereafter, some classes of ordered semigroups are characterized in terms of FBS sets and hence a few characterization theorems are

characterized by means of their FBS quasi-ideals while the ordered semigroups that are both intra-regular and left weakly-regular are characterized in terms of their FBS left (right) and FBS quasi-ideals. Finally, left and right simple (resp., completely regular) ordered semigroups are characterized by their FBS quasi-ideals [resp., FBS (semiprime) quasi-ideals]. It is proved that S is completely regular iff every FBS quasi-ideal λ_A over S is FBS semiprime quasi-ideal over S.

2. Preliminaries

In this section, we recall some basic definitions that are helpful in the comprehension of the content of the article.

An ordered semigroup $(S,*,\leq)$ is a set S, where $(S,*)$ is a semigroup and $(S,\leq)$ is a partially ordered set (poset) such that the order relation "$\leq$" is compatible with the binary operation "$*$" on S. Likewise, a non-empty subset $\mathcal{A}$ of an ordered semigroup $(S,\cdot,\leq)$ is called a subsemigroup of S iff (i) $\mathcal{A}\mathcal{A}\subseteq\mathcal{A}$ and (ii) $(\mathcal{A},\cdot,\leq)$ is an ordered semigroup. For a nonempty subset P of an ordered semigroup S, we denote by $(P]$ the subset of S defined as

$$(P]=\{\sigma\in S\mid \sigma\leq p \text{ for some } p\in P\}.$$

Similarly, for $\sigma\in S$, we denote by X_σ the subset of $S\times S$ defined as follows:

$$X_\sigma \quad = \quad \{(a,b)\in S\times S\mid \sigma\leq ab\}.$$

Naz and Shabir [28] presented the notion of FBS sets and explained the application of the structure in decision making problems through an example. The authors redefined the notion and studied FBS (left, right) ideals [3], FBS semiprime ideals [4] and FBS bi-ideals [5] in ordered semigroup theory. Let us recall the concept of FBS set as follows:

Definition 1. *[3] Assume $\mathcal{S}$ be an initial universe set, $\mathcal{F}(\mathcal{S})$ be the collection of all fuzzy subsets of $\mathcal{S}$ and E be a set of parameters. For $A\subseteq E$, let $f:A\to E$ be an injective* *function. Then, an FBS set λ_A over $\mathcal{S}$ is an object of the form*

$$\lambda_A = (\overset{+}{\lambda}, \overset{-}{\lambda}, A),$$

where $\overset{+}{\lambda} : A \to \mathcal{F}(\mathcal{S})$ and $\overset{-}{\lambda} : f(A) \to \mathcal{F}(\mathcal{S})$ are set-valued functions such that the condition

$$0 \leq \overset{+}{\lambda}(\varepsilon)(x) + \overset{-}{\lambda}(f(\varepsilon))(x) \leq 1$$

holds, for all $x \in \mathcal{S}$ and $\varepsilon \in A$.

For any $\varepsilon \in A$, the degree $\overset{+}{\lambda}(\varepsilon)(x)$ characterizes the extent that the element $x \in \mathcal{S}$ satisfies the property of the corresponding fuzzy set, whereas the degree $\overset{-}{\lambda}(f(\varepsilon))(x)$ indicates the extent that x satisfies the counter-property. Besides, for all $\varepsilon \in A$ and $x \in \mathcal{S}$, the condition

$$0 \leq \overset{+}{\lambda}(\varepsilon)(x) + \overset{-}{\lambda}(f(\varepsilon))(x) \leq 1$$

is imposed as consistency constraint. The FBS set λ_A can also be represented as

$$\lambda_A = \{(\varepsilon, \overset{+}{\lambda}(\varepsilon), \overset{-}{\lambda}(f(\varepsilon))) \mid \varepsilon \in A\}.$$

If $\overset{+}{\lambda}(\varepsilon) = \phi$, the empty fuzzy set of $\mathcal{S}$ and $\overset{-}{\lambda}(f(\varepsilon)) = \mathcal{S}$, the universal fuzzy set of $\mathcal{S}$ for any $\varepsilon \in A$, then the element $(\varepsilon, \phi, \mathcal{S})$ will not appear in λ_A.

For the sake of brevity, we shall denote $\overset{-}{\lambda}(f(\varepsilon))$ by $\overset{-}{\lambda}(\varepsilon)$ and, similarly, write $(\overset{+}{\lambda}, \overset{-}{\lambda})$ instead of $(\overset{+}{\lambda}, \overset{-}{\lambda}, A)$.

Definition 2. *[3] Let λ_A be an FBS set over $\mathcal{S}$ such that, for all $\varepsilon \in A$, we have $\overset{+}{\lambda}(\varepsilon) = \mathcal{S}$, the universal fuzzy set of $\mathcal{S}$ and $\overset{-}{\lambda}(\varepsilon) = \phi$, the null fuzzy set of $\mathcal{S}$. Then λ_A is called universal FBS set over $\mathcal{S}$. We denote it by $\mathcal{S}_A = (\overset{+}{\mathcal{S}}, \overset{-}{\mathcal{S}})$. Likewise if, for all $\varepsilon \in A$, we have $\overset{+}{\lambda}(\varepsilon) = \phi$, the null fuzzy set of S and $\overset{-}{\lambda}(\varepsilon) = S$, the universal fuzzy set of S, then λ_A is called null FBS set over S. We denote it by $\Phi_A = (\overset{+}{\Phi}, \overset{-}{\Phi})$.*

Definition 3. *[3] Let λ_A and δ_A be FBS sets over $\mathcal{S}$. We say that λ_A is an FBS subset of δ_A, denoted as $\lambda_A \tilde{\preceq} \delta_A$, iff $\overset{+}{\lambda} \leq \overset{+}{\delta}$ and $\overset{-}{\delta} \leq \overset{-}{\lambda}$ iff $\overset{+}{\lambda}(\varepsilon)(x) \leq \overset{+}{\delta}(\varepsilon)(x)$ and $\overset{-}{\delta}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(x)$, for all $x \in \mathcal{S}$ and $\varepsilon \in A$. Similarly, λ_A is said to be an FBS superset of δ_A iff δ_A is an FBS subset of λ_A.*

Definition 4. *[3] For two FBS sets λ_A and δ_A over $\mathcal{S}$, we say that λ_A and δ_A are FBS equal iff $\overset{+}{\lambda} = \overset{+}{\delta}$ and $\overset{-}{\lambda} = \overset{-}{\delta}$. This relationship is denoted by $\lambda_A = \delta_A$. Further, we note that $\overset{+}{\lambda} = \overset{+}{\delta}$ and $\overset{-}{\lambda} = \overset{-}{\delta}$ iff $\overset{+}{\lambda}(\varepsilon)(x) = \overset{+}{\delta}(\varepsilon)(x)$ and $\overset{-}{\lambda}(\varepsilon)(x) = \overset{-}{\delta}(\varepsilon)(x)$ for all $x \in \mathcal{S}$ and $\varepsilon \in A$. Equivalently, λ_A and δ_A are FBS equal iff*

$$\lambda_A \tilde{\preceq} \delta_A \quad \textit{and} \quad \delta_A \tilde{\preceq} \lambda_A.$$

Definition 5. *[3] Let λ_A and δ_A be FBS sets over $\mathcal{S}$. The FBS intersection of the two FBS sets is an FBS set γ_A over $\mathcal{S}$ defined by*

$$\overset{+}{\gamma}(\varepsilon)(\sigma) = (\overset{+}{\lambda} \wedge \overset{+}{\delta})(\varepsilon)(\sigma) = \min\{\overset{+}{\lambda}(\varepsilon)(\sigma), \overset{+}{\delta}(\varepsilon)(\sigma)\}$$

and

$$\overset{-}{\gamma}(\varepsilon)(\sigma) = (\overset{-}{\lambda} \vee \overset{-}{\delta})(\varepsilon)(\sigma) = \max\{\overset{-}{\lambda}(\varepsilon)(\sigma), \overset{-}{\delta}(\varepsilon)(\sigma)\}$$

for all $\sigma \in \mathcal{S}$ and $\varepsilon \in A$. We denote $\gamma_A = \lambda_A \tilde{\cap} \delta_A$, where $\overset{+}{\gamma} = \overset{+}{\lambda} \wedge \overset{+}{\delta}$ and $\overset{-}{\gamma} = \overset{-}{\lambda} \vee \overset{-}{\delta}$. Here, the symbols $\wedge$ and $\vee$ respectively represent the operations of fuzzy intersection and fuzzy union of two fuzzy sets.

Definition 6. *[3] Let λ_A and δ_A be FBS sets over $\mathcal{S}$. The FBS union of the two FBS sets is an FBS set γ_A over $\mathcal{S}$ defined by*

$$\overset{+}{\gamma}(\varepsilon)(\sigma) = (\overset{+}{\lambda} \vee \overset{+}{\delta})(\varepsilon)(\sigma) = \max\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\delta}(\varepsilon)(x)\}$$

and

$$\overset{-}{\gamma}(\varepsilon)(\sigma) = (\overset{-}{\lambda} \wedge \overset{-}{\lambda})(\varepsilon)(\sigma) = \min\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\delta}(\varepsilon)(x)\}$$

for all $\sigma \in \mathcal{S}$ and $\varepsilon \in A$. We denote $\gamma_A = \lambda_A \tilde{\cup} \delta_A$, where $\overset{+}{\gamma} = \overset{+}{\lambda} \vee \overset{+}{\delta}$ and $\overset{-}{\gamma} = \overset{-}{\lambda} \wedge \overset{-}{\delta}$.

In what follows, S always represents an ordered semigroup.

Definition 7. *[3] Let λ_A be an FBS set over S such that, for any $x, y \in S$, if $x \leq y$ then $\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y)$ and $\overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y)$ for all $\varepsilon \in A$. We say that λ_A is an FBS ordered semigroup over S iff, for all $\varepsilon \in A$ and $x, y \in S$, the following assertions hold:*

(i) $\overset{+}{\lambda}(\varepsilon)(xy) \geq \min\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)\}$.

(ii) $\overset{-}{\lambda}(\varepsilon)(xy) \leq \max\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(y)\}$.

Definition 8. *[3, 4] Let λ_A be an FBS sets over S. For $\epsilon \in A$ and $r \in (0, 1]$, $t \in [0, 1)$, we denote by $\overset{(r,t)}{\lambda_A}(\epsilon)$ the subset of S that is defined as follows:*

$$\overset{(r,t)}{\lambda_A}(\epsilon) = \{x \in S : \overset{+}{\lambda}(\epsilon)(x) \geq r, \ \overset{-}{\lambda}(\epsilon)(x) \leq t\}.$$

For any $\epsilon \in A$, the subset $\overset{(r,t)}{\lambda_A}(\epsilon)$ of S is called an (r, t)-level subset of λ_A.

Definition 9. *[3] Let $\varGamma_A$ and $\varUpsilon_A$ be FBS sets over S. Let $p, q, \sigma \in S$ and $\varepsilon \in A$. Then, the product of $\varGamma_A$ and $\varUpsilon_A$ is defined to be the FBS set γ_A over S, where*

$$\overset{+}{\gamma}(\varepsilon)(\sigma) = \begin{cases} \bigvee_{(p,q) \in X_\sigma} \min\{\overset{+}{\varGamma}(\varepsilon)(p), \overset{+}{\varUpsilon}(\varepsilon)(q)\} & \text{if } X_\sigma \neq \phi, \\ 0 & \text{if } X_\sigma = \phi, \end{cases}$$

and

$$\bar{\gamma}(\varepsilon)(\sigma) = \begin{cases} \bigwedge_{(p,q)\in X_\sigma} \max\{\bar{\Gamma}(\varepsilon)(p), \bar{\Upsilon}(\varepsilon)(q)\} & \text{if } X_\sigma \neq \phi, \\ 1 & \text{if } X_\sigma = \phi. \end{cases}$$

We denote $\gamma_A = \Gamma_A \circ \Upsilon_A$, *where* $\overset{+}{\gamma} = \overset{+}{\Gamma} \circ \overset{+}{\Upsilon}$ *and* $\bar{\gamma} = \bar{\Gamma} \circ \bar{\Upsilon}$.

Definition 10. *[3, 4] Let λ_A be an FBS set over S such that, for any $x, y \in S$, if $x \leq y$ then $\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y)$ and $\bar{\lambda}(\varepsilon)(x) \leq \bar{\lambda}(\varepsilon)(y)$ for all $\varepsilon \in A$. We say that λ_A is an FBS left ideal over S iff, for all $x, y \in S$ and $\varepsilon \in A$, the following assertions hold:*

(i) $\overset{+}{\lambda}(\varepsilon)(xy) \geq \overset{+}{\lambda}(\varepsilon)(y)$.
(ii) $\bar{\lambda}(\varepsilon)(xy) \leq \bar{\lambda}(\varepsilon)(y)$.

Dually, one can define an FBS right ideal over S.

Definition 11. *[3, 4] An FBS set λ_A over S is called an FBS two-sided ideal or, simply, an FBS ideal over S if it is both an FBS left ideal and an FBS right ideal over S. Equivalently, λ_A is an FBS ideal over S iff, for all $x, y \in S$ and $\varepsilon \in A$, the following assertions are satisfied:*

(i) $\overset{+}{\lambda}(\varepsilon)(xy) \geq \max\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)\}$.
(ii) $\bar{\lambda}(\varepsilon)(xy) \leq \min\{\bar{\lambda}(\varepsilon)(x), \bar{\lambda}(\varepsilon)(y)\}$.

Definition 12. *[3] Let $\mathcal{P}$ be a non-empty subset of S. Then an FBS set over S of the form*

$$\overset{\mathcal{P}}{\chi}_A = (\overset{+}{\chi}_{\mathcal{P}}, \bar{\chi}_{\mathcal{P}}, A)$$

is called an FBS characteristic function of $\mathcal{P}$, where

$$\overset{+}{\chi}_{\mathcal{P}}(\epsilon)(\sigma) = \begin{cases} 1 & \text{if } \sigma \in \mathcal{P}, \\ 0 & \text{if } \sigma \notin \mathcal{P}, \end{cases}$$

and

$$\bar{\chi}_{\mathcal{P}}(\epsilon)(\sigma) = \begin{cases} 0 & \text{if } \sigma \in \mathcal{P}, \\ 1 & \text{if } \sigma \notin \mathcal{P}, \end{cases}$$

for all $\epsilon \in A$ and $\sigma \in S$.

3. THE CONCEPT OF FUZZY BIPOLAR SOFT QUASI-IDEALS IN ORDERED SEMIGROUPS

In this section, we introduce the concept of an FBS quasi-ideal in ordered semigroup theory. The concept is elaborated by a suitable example.

Definition 13. *A non-empty subset $\mathcal{Q}$ of S is called a quasi-ideal of S iff the following axioms are satisfied:*

(1) $(\mathcal{Q}S] \cap (S\mathcal{Q}] \subseteq \mathcal{Q}$, *and*
(2) *If $a \in \mathcal{Q}$ and $S \ni b \leq a$, then $b \in \mathcal{Q}$.*

Definition 14. *Let λ_A be an FBS set over S and $x, y \in S$. Then, it is called an FBS quasi-ideal over S iff, for all $\varepsilon \in A$, the following axioms are satisfied:*

(i) *If $x \leq y$, then $\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y)$ and $\overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y)$.*

(ii) $(\lambda_A \circ S_A) \,\widetilde{\cap}\, (S_A \circ \lambda_A) \,\widetilde{\preceq}\, \lambda_A$, *that is,*

$$(\overset{+}{\lambda} \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\lambda}) \leq \overset{+}{\lambda}, \quad (\overset{-}{\lambda} \circ \overset{-}{S}) \vee (\overset{-}{S} \circ \overset{-}{\lambda}) \geq \overset{-}{\lambda}.$$

Example 1. *Consider the ordered semigroup $S = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$ with the multiplication "$\cdot$" and the order relation "$\leq$" given below:*

$\cdot$	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6
γ_0	γ_0	γ_0	γ_0	γ_0	γ_0	γ_0	γ_0
γ_1	γ_0	γ_1	γ_0	γ_0	γ_4	γ_0	γ_0
γ_2	γ_0	γ_0	γ_2	γ_0	γ_0	γ_0	γ_0
γ_3	γ_0	γ_6	γ_0	γ_3	γ_3	γ_0	γ_6
γ_4	γ_0	γ_1	γ_0	γ_4	γ_4	γ_0	γ_1
γ_5	γ_0	γ_0	γ_0	γ_0	γ_0	γ_5	γ_0
γ_6	γ_0	γ_6	γ_0	γ_0	γ_3	γ_0	γ_0

$$\leq = \{(\gamma_0, \gamma_0), (\gamma_1, \gamma_1), (\gamma_2, \gamma_2), (\gamma_3, \gamma_3), (\gamma_4, \gamma_4), (\gamma_5, \gamma_5), (\gamma_6, \gamma_6)\}.$$

Let $A = \{0, 1\} = E = Z_2$ be a set of parameters and $f : A \to Z_2$ be defined by $f(\epsilon) = \epsilon^{-1}$ for all $\epsilon \in A$. Let λ_A be an FBS set over S which is defined in terms of its fuzzy approximate functions such that

$$\overset{+}{\lambda}(0)(x) = \begin{cases} 0.6 & \text{if } x \in \{\gamma_0, \gamma_2\}, \\ 0.5 & \text{if } x \in \{\gamma_1, \gamma_5\}, \\ 0.4 & \text{if } x = \gamma_4, \\ 0.3 & \text{if } x \in \{\gamma_3, \gamma_6\}, \end{cases}$$

$$\overset{-}{\lambda}(0)(x) = \begin{cases} 0.3 & \text{if } x \in \{\gamma_0, \gamma_2\}, \\ 0.4 & \text{if } x \in \{\gamma_1, \gamma_5\}, \\ 0.5 & \text{if } x = \gamma_4, \\ 0.6 & \text{if } x \in \{\gamma_3, \gamma_6\}, \end{cases}$$

$$\overset{+}{\lambda}(1)(x) = \begin{cases} 0.5 & \text{if } x \in \{\gamma_0, \gamma_5\}, \\ 0.4 & \text{if } x \in \{\gamma_1, \gamma_2\}, \\ 0.3 & \text{if } x = \gamma_4, \\ 0.2 & \text{if } x \in \{\gamma_3, \gamma_6\}, \end{cases}$$

and

$$\overset{-}{\lambda}(1)(x) = \begin{cases} 0.2 & \text{if } x \in \{\gamma_0, \gamma_5\}, \\ 0.3 & \text{if } x \in \{\gamma_1, \gamma_2\}, \\ 0.4 & \text{if } x = \gamma_4, \\ 0.5 & \text{if } x \in \{\gamma_3, \gamma_6\}, \end{cases}$$

Then one can check by routine calculations that λ_A is an FBS quasi-ideal over S.

4. Relationship between crisp quasi-ideals of ordered semigroups and thir fuzzy bipolar soft quasi-ideals

In this section, a relationship between crisp quasi-ideals of ordered semigroups and the FBS quasi-ideals is investigated by using the concept of (r,t)-level subset (resp., FBS characteristic function). In this connection, it is proved that an FBS set λ_A over S is an FBS quasi-ideal over S if and only if the (r,t)-level subset $\overset{(r,t)}{\lambda_A}(\varepsilon) \neq \phi$ of λ_A is a quasi-ideal of S for all $r \in (0,1],\ t \in [0,1)$ and $\varepsilon \in A$. Likewise, it is revealed that a non-empty subset Q of S is a quasi-ideal of S if and only if the FBS characteristic function $\overset{Q}{\chi_A}$ of Q is an FBS quasi-ideal over S.

Theorem 1. *An FBS set λ_A over S is an FBS quasi-ideal over S if and only if $\overset{(r,t)}{\lambda_A}(\varepsilon) \neq \phi$ is a quasi-ideal of S for all $r \in (0,1],\ t \in [0,1)$ and $\varepsilon \in A$.*

Proof. It is straightforward. □

As an application of Theorem 1, we present the following example:

Example 2. *Consider Example 1, where λ_A has been defined to be an FBS quasi-ideal over the ordered semigroup $S = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$. Let us define the (r,t)-level subsets $\overset{(r,t)}{\lambda_A}(0)$ and $\overset{(r,t)}{\lambda_A}(1)$ of λ_A as follows:*

$$\overset{(r,t)}{\lambda_A}(0) = \begin{cases} S & \text{if } r \in (0, 0.3], \\ \{\gamma_0, \gamma_1, \gamma_2, \gamma_4, \gamma_5\} & \text{if } r \in (0.3, 0.4], \\ \{\gamma_0, \gamma_1, \gamma_2, \gamma_5\} & \text{if } r \in (0.4, 0.5], \\ \{\gamma_0, \gamma_2\} & \text{if } r \in (0.5, 0.6], \\ \phi & \text{if } r \in (0.6, 1], \\ \phi & \text{if } t \in [0, 0.3), \\ \{\gamma_0, \gamma_2\} & \text{if } t \in [0.3, 0.4), \\ \{\gamma_0, \gamma_1, \gamma_2, \gamma_5\} & \text{if } t \in [0.4, 0.5), \\ \{\gamma_0, \gamma_1, \gamma_2, \gamma_4, \gamma_5\} & \text{if } t \in [0.5, 0.6), \\ S & \text{if } t \in [0.6, 1), \end{cases}$$

and

$$\overset{(r,t)}{\lambda_A}(1) = \begin{cases} S & \text{if } r \in (0, 0.2], \\ \{\gamma_0, \gamma_1, \gamma_2, \gamma_4, \gamma_5\} & \text{if } r \in (0.2, 0.3], \\ \{\gamma_0, \gamma_1, \gamma_2, \gamma_5\} & \text{if } r \in (0.3, 0.4], \\ \{\gamma_0, \gamma_5\} & \text{if } r \in (0.4, 0.5], \\ \phi & \text{if } r \in (0.5, 1], \\ \phi & \text{if } t \in [0, 0.2), \\ \{\gamma_0, \gamma_5\} & \text{if } t \in [0.2, 0.3), \\ \{\gamma_0, \gamma_1, \gamma_2, \gamma_5\} & \text{if } t \in [0.3, 0.4), \\ \{\gamma_0, \gamma_1, \gamma_2, \gamma_4, \gamma_5\} & \text{if } t \in [0.4, 0.5), \\ S & \text{if } t \in [0.5, 1). \end{cases}$$

We note that, for all $r \in (0, 1]$, $t \in [0, 1)$ and $0, 1 \in A$, the (r, t)-level subsets $\overset{(r,t)}{\lambda_A}(0) \neq \phi$ and $\overset{(r,t)}{\lambda_A}(1) \neq \phi$ of λ_A are quasi-ideals of S. Then, by Theorem 1, we have λ_A is an FBS quasi-ideal over S. Thus, as an application of Theorem 1, it is verified that the FBS quasi-ideal over S defined in Example 1 is, in fact, an FBS quasi-ideal over S.

Before establishing the main theorem that characterizes ordinary quasi-ideals of ordered semigroups by the FBS quasi-ideals using the concept of FBS characteristic function, we introduce the following results:

Proposition 4.1. *Let P and Q be nonempty subsets of S. Let $\overset{P}{\chi_A}$ and $\overset{Q}{\chi_A}$ be FBS characteristic functions of P and Q, respectively. Then $P \subseteq Q$ if and only if*

$$\overset{+}{\chi}_P \leq \overset{+}{\chi}_Q, \quad \overset{-}{\chi}_P \geq \overset{-}{\chi}_Q.$$

Proof. It is straightforward. □

Proposition 4.2. *Let P and Q be nonempty subsets of S. Let $\overset{P}{\chi_A}$ and $\overset{Q}{\chi_A}$ be FBS characteristic functions of P and Q, respectively. Then, we have*

$$\overset{+}{\chi}_P \wedge \overset{+}{\chi}_Q = \overset{+}{\chi}_{P \cap Q}, \quad \overset{-}{\chi}_P \vee \overset{-}{\chi}_Q = \overset{-}{\chi}_{P \cap Q}.$$

Proof. It is straightforward. □

Lemma 4.3. *Let P and Q be nonempty subsets of S. Let $\overset{P}{\chi_A}$ and $\overset{Q}{\chi_A}$ be FBS characteristic functions of P and Q, respectively. Then, we have*

$$(\overset{+}{\chi}_P \circ \overset{+}{\chi}_Q) = \overset{+}{\chi}_{(PQ]}, \quad (\overset{-}{\chi}_P \circ \overset{-}{\chi}_Q) = \overset{-}{\chi}_{(PQ]}.$$

Proof. It is straightforward. □

The following theorem characterizes crisp (or, ordinary) quasi-ideals of an ordered semigroup S by the FBS quasi-ideals over S.

Theorem 2. *Let $\phi \neq Q \subseteq S$. Then Q is a quasi-ideal of S if and only if the FBS characteristic function $\overset{Q}{\chi}_A$ of Q is an FBS quasi-ideal over S.*

Proof. First assume that Q be a quasi-ideal of S. Let $\overset{Q}{\chi}_A$ be the FBS characteristic function of Q. Then, by virtue of Lemma 4.3, Proposition 4.2 and the fact that $S_A = \overset{S}{\chi}_A$, we have

$$\begin{aligned}(\overset{+}{\chi}_Q \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\chi}_Q) &= (\overset{+}{\chi}_Q \circ \overset{+}{\chi}_S) \wedge (\overset{+}{\chi}_S \circ \overset{+}{\chi}_Q) \\ &= \overset{+}{\chi}_{(QS]} \wedge \overset{+}{\chi}_{(SQ]} \\ &= \overset{+}{\chi}_{(QS] \cap (SQ]}\end{aligned}$$

and

$$\begin{aligned}(\overset{-}{\chi}_Q \circ \overset{-}{S}) \vee (\overset{-}{S} \circ \overset{-}{\chi}_Q) &= (\overset{-}{\chi}_Q \circ \overset{-}{\chi}_S) \vee (\overset{-}{\chi}_S \circ \overset{-}{\chi}_Q) \\ &= \overset{-}{\chi}_{(QS]} \vee \overset{-}{\chi}_{(SQ]} \\ &= \overset{-}{\chi}_{(QS] \cap (SQ]}.\end{aligned}$$

Since $(QS] \cap (SQ] \subseteq Q$, thus, by Proposition 4.1, we have $\overset{+}{\chi}_{(QS] \cap (SQ]} \leq \overset{+}{\chi}_Q$ and $\overset{-}{\chi}_{(QS] \cap (SQ]} \geq \overset{-}{\chi}_Q$. Then, it follows that

$$(\overset{+}{\chi}_Q \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\chi}_Q) \leq \overset{+}{\chi}_Q$$

and

$$(\overset{-}{\chi}_Q \circ \overset{-}{S}) \vee (\overset{-}{S} \circ \overset{-}{\chi}_Q) \geq \overset{-}{\chi}_Q.$$

Now, let $x, y \in S$ such that $x \leq y$. If $y \notin Q$, then $\overset{+}{\chi}_Q(\varepsilon)(y) = 0$ and $\overset{-}{\chi}_Q(\varepsilon)(y) = 1$. Thus, it follows that

$$\overset{+}{\chi}_Q(\varepsilon)(x) \geq \overset{+}{\chi}_Q(\varepsilon)(y), \quad \overset{-}{\chi}_Q(\varepsilon)(x) \leq \overset{-}{\chi}_Q(\varepsilon)(y).$$

If $y \in Q$, then $\overset{+}{\chi}_Q(\varepsilon)(y) = 1$ and $\overset{-}{\chi}_Q(\varepsilon)(y) = 0$. Further, since $y \in Q$, thus $x \in Q$. Then, we have $\overset{+}{\chi}_Q(\varepsilon)(x) = 1$ and $\overset{-}{\chi}_Q(\varepsilon)(x) = 0$. Thus, once more, it follows that

$$\overset{+}{\chi}_Q(\varepsilon)(x) \geq \overset{+}{\chi}_Q(\varepsilon)(y), \quad \overset{-}{\chi}_Q(\varepsilon)(x) \leq \overset{-}{\chi}_Q(\varepsilon)(y).$$

Therefore, we conclude that $\overset{Q}{\chi}_A$ is an FBS quasi-ideal over S.

Conversely, assume that the FBS characteristic function $\overset{Q}{\chi}_A$ of Q is an FBS quasi-ideal over S. Then, since $S_A = \overset{S}{\chi}_A$, we have

$$\begin{aligned}\overset{+}{\chi}_Q &\geq (\overset{+}{\chi}_Q \circ \overset{+}{\chi}_S) \wedge (\overset{+}{\chi}_S \circ \overset{+}{\chi}_Q) \\ &= (\overset{+}{\chi}_Q \circ \overset{+}{\chi}_S) \wedge (\overset{+}{\chi}_S \circ \overset{+}{\chi}_Q) \\ &= \overset{+}{\chi}_{(QS]} \wedge \overset{+}{\chi}_{(SQ]} \\ &= \overset{+}{\chi}_{(QS] \cap (SQ]}\end{aligned}$$

and, similarly,

$$\overset{-}{\chi}_Q \quad \leq \quad \overset{-}{\chi}_{(QS]\cap(SQ]}.$$

Then, by Proposition 4.1, it follows that $(QS]\cap(SQ] \subseteq Q$. Now, let $x \in S$ and $x \leq y \in Q$. Then, since $\overset{Q}{\chi}_A$ is an FBS quasi-ideal over S, we have

$$\overset{+}{\chi}_Q(\varepsilon)(x) \geq \overset{+}{\chi}_Q(\varepsilon)(y),$$

$$\overset{-}{\chi}_Q(\varepsilon)(x) \leq \overset{-}{\chi}_Q(\varepsilon)(y).$$

Moreover, since $y \in Q$, we have $\overset{+}{\chi}_Q(\varepsilon)(y) = 1$ and $\overset{-}{\chi}_Q(\varepsilon)(y) = 0$. Thus, it follows that $\overset{+}{\chi}_Q(\varepsilon)(x) = 1$ and $\overset{-}{\chi}_Q(\varepsilon)(x) = 0$ which implies that $x \in Q$. Therefore, Q is a quasi-ideal of S. □

In the following section, we further discuss some properties of FBS quasi-ideals over ordered semigroups.

5. Some miscellaneous characteristics of fuzzy bipolar soft quasi-ideals in ordered semigroups

In this section, we examine some miscellaneous properties of FBS quasi-ideals over S. It is proved that every FBS quasi-ideal over S is an FBS ordered semigroup. In addition, it is revealed that every FBS (left, right) ideal λ_A over S is an FBS quasi-ideal, however, by a counterexample, it is exposed that the converse of the assertion is not true in general. Moreover, it is proved that every FBS quasi-ideal λ_A over S is an FBS bi-ideal, yet, by a counterexample, it is revealed that every FBS bi-ideal λ_A over S is not necessarily an FBS quasi-ideal. It is interesting to note that the concepts of FBS bi-ideal and FBS quasi-ideal in regular ordered semigroups coincide.

An ordered semigroup S is regular iff, for every $\sigma \in S$, there exists $x \in S$ such that $\sigma \leq \sigma x \sigma$. Equivalently, we have (i) S is regular iff $\sigma \in (\sigma S \sigma]$ for all $\sigma \in S$ and (ii) S is regular iff $A \subseteq (ASA] \quad \forall A \subseteq S$ [8].

Proposition 5.1. *Let λ_A and δ_A be FBS sets over S. Then, the following assertions are satisfied:*

(i) $\lambda_A \mathbin{\widetilde{\cap}} \delta_A \mathrel{\widetilde{\preceq}} \lambda_A, \quad \lambda_A \mathbin{\widetilde{\cap}} \delta_A \mathrel{\widetilde{\preceq}} \delta_A$.
(ii) $\lambda_A \mathrel{\widetilde{\preceq}} \lambda_A \mathbin{\widetilde{\cup}} \delta_A, \quad \delta_A \mathrel{\widetilde{\preceq}} \lambda_A \mathbin{\widetilde{\cup}} \delta_A$.

Proof. It is straightforward. □

Proposition 5.2. *Let λ_A, δ_A, γ_A and ϑ_A be FBS sets over S such that we have $\lambda_A \mathrel{\widetilde{\preceq}} \delta_A$ and $\gamma_A \mathrel{\widetilde{\preceq}} \vartheta_A$. Then the following assertions are satisfied:*

(i) $\lambda_A \mathbin{\widetilde{\cap}} \gamma_A \mathrel{\widetilde{\preceq}} \delta_A \mathbin{\widetilde{\cap}} \vartheta_A$.
(ii) $\lambda_A \mathbin{\widetilde{\cup}} \gamma_A \mathrel{\widetilde{\preceq}} \delta_A \mathbin{\widetilde{\cup}} \vartheta_A$.

Proof. It is straightforward. □

Proposition 5.3. *Every FBS quasi-ideal over S is an FBS ordered semigroup.*

Proof. It is straightforward. □

Proposition 5.4. *Let $a \leq a^2$ for all $a \in S$. Then, for every FBS quasi-ideal λ_A over S, we have*

$$\overset{+}{\lambda}(\varepsilon)(a) = \overset{+}{\lambda}(\varepsilon)(a^2), \quad \overset{-}{\lambda}(\varepsilon)(a) = \overset{-}{\lambda}(\varepsilon)(a^2)$$

for all $\varepsilon \in A$.

Proof. Let λ_A be an FBS quasi-ideal over S. Further, let $\varepsilon \in A$ and $a \in S$. Then, by Proposition 5.3, we have λ_A is an FBS ordered semigroup over S. Moreover, since $a \leq a^2$, we have

$$\overset{+}{\lambda}(\varepsilon)(a) \quad \geq \quad \overset{+}{\lambda}(\varepsilon)(a^2) \geq \min\{\overset{+}{\lambda}(\varepsilon)(a), \overset{+}{\lambda}(\varepsilon)(a)\} = \overset{+}{\lambda}(\varepsilon)(a)$$

and

$$\overset{-}{\lambda}(\varepsilon)(a) \quad \leq \quad \overset{-}{\lambda}(\varepsilon)(a^2) \leq \max\{\overset{-}{\lambda}(\varepsilon)(a), \overset{-}{\lambda}(\varepsilon)(a)\} = \overset{-}{\lambda}(\varepsilon)(a).$$

This completes the proof. □

Proposition 5.5. *Every FBS right ideal over S is an FBS quasi-ideal.*

Proof. Let λ_A be an FBS right ideal over S. Let $\varepsilon \in A$ and $a \in S$. If $X_a = \phi$, then

$$(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(a) \quad = \quad 0 = (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(a)$$

and

$$(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(a) \quad = \quad 1 = (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(a).$$

So, we have

$$\begin{aligned} ((\overset{+}{\lambda} \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\lambda}))(\varepsilon)(a) \quad &= \quad \min\{(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(a), (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(a)\} \\ &= \quad 0 \leq \overset{+}{\lambda}(\varepsilon)(a) \end{aligned}$$

and

$$\begin{aligned} ((\overset{-}{\lambda} \circ \overset{-}{S}) \vee (\overset{-}{S} \circ \overset{-}{\lambda}))(\varepsilon)(a) \quad &= \quad \max\{(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(a), (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(a)\} \\ &= \quad 1 \geq \overset{-}{\lambda}(\varepsilon)(a). \end{aligned}$$

Let $X_a \neq \phi$. Then, we have

$$(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(a) \quad = \quad \bigvee_{(p,q) \in X_a} \min\{\overset{+}{\lambda}(\varepsilon)(p), \overset{+}{S}(\varepsilon)(q)\}$$

and

$$(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(a) \quad = \quad \bigwedge_{(p,q) \in X_a} \max\{\overset{-}{\lambda}(\varepsilon)(p), \overset{-}{S}(\varepsilon)(q)\}.$$

On the other hand, if $(p, q) \in X_a$, then $a \leq pq$. So, we have

$$\overset{+}{\lambda}(\varepsilon)(a) \geq \overset{+}{\lambda}(\varepsilon)(pq) \geq \overset{+}{\lambda}(\varepsilon)(p) = \min\{\overset{+}{\lambda}(\varepsilon)(p), \overset{+}{S}(\varepsilon)(q)\}$$

and

$$\overset{-}{\lambda}(\varepsilon)(a) \leq \overset{-}{\lambda}(\varepsilon)(pq) \leq \overset{-}{\lambda}(\varepsilon)(p) = \max\{\overset{-}{\lambda}(\varepsilon)(p), \overset{-}{S}(\varepsilon)(q)\}.$$

So, it follows that

$$\begin{aligned}
\overset{+}{\lambda}(\varepsilon)(a) &\geq \bigvee_{(p,q)\in X_a} \min\{\overset{+}{\lambda}(\varepsilon)(p), \overset{+}{S}(\varepsilon)(q)\} \\
&= (\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(a) \\
&\geq \min\{(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(a), (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(a)\} \\
&= ((\overset{+}{\lambda} \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\lambda}))(\varepsilon)(a)
\end{aligned}$$

and, similarly,

$$\overset{-}{\lambda}(\varepsilon)(a) \leq ((\overset{-}{\lambda} \circ \overset{-}{S}) \vee (\overset{-}{S} \circ \overset{-}{\lambda}))(\varepsilon)(a).$$

Thus, we obtain

$$(\overset{+}{\lambda} \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\lambda}) \leq \overset{+}{\lambda}$$

and

$$(\overset{-}{\lambda} \circ \overset{-}{S}) \vee (\overset{-}{S} \circ \overset{-}{\lambda}) \geq \overset{-}{\lambda}.$$

Therefore, λ_A is an FBS quasi-ideal over S. □

Similarly, one can prove the following proposition:

Proposition 5.6. *Every FBS left ideal over S is an FBS quasi-ideal.*

In the light of Propositions 5.5 and 5.6, we establish the following result:

Proposition 5.7. *Every FBS two-sided ideal over S is an FBS quasi-ideal.*

The converse of Proposition 5.7 is not true in general. Thus, we have the following assertion:

Remark 1. *Every FBS quasi-ideal over S is not necessarily an FBS ideal over S.*

Lemma 5.8. *An FBS set λ_A over S is an FBS two-sided ideal (resp., FBS left ideal, FBS right ideal) over S if and only if $\overset{(r,t)}{\lambda_A}(\varepsilon) \neq \phi$ is a two-sided ideal (resp., left ideal, right ideal) of S for all $r \in (0, 1],\ t \in [0, 1)$ and $\varepsilon \in A$.*

Proof. It is straightforward. □

To justify Remark 1, we present the following example:

Example 3. *Consider the ordered semigroup $S = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ with the multiplication "$\cdot$" and the order relation "$\leq$" given as follows:*

$\cdot$	γ_0	γ_1	γ_2	γ_3	γ_4
γ_0	γ_0	γ_0	γ_0	γ_0	γ_0
γ_1	γ_0	γ_1	γ_2	γ_0	γ_0
γ_2	γ_0	γ_0	γ_0	γ_0	γ_0
γ_3	γ_0	γ_0	γ_0	γ_3	γ_0
γ_4	γ_0	γ_4	γ_0	γ_0	γ_0

$$\leq = \{(\gamma_0,\gamma_0),(\gamma_0,\gamma_1),(\gamma_0,\gamma_2),(\gamma_0,\gamma_4),(\gamma_1,\gamma_1),(\gamma_2,\gamma_2),(\gamma_3,\gamma_3),(\gamma_4,\gamma_4)\}.$$

Let $E = S_3 = \{\epsilon_0 = (\ 1\ \ 2\ \ 3\ ), \epsilon_1 = (\ 3\ \ 1\ \ 2\ ), \epsilon_2 = (\ 2\ \ 3\ \ 1\ ), \epsilon_3 = (\ 2\ \ 1\ \ 3\ ), \epsilon_4 = (\ 3\ \ 2\ \ 1\ ), \epsilon_5 = (\ 1\ \ 3\ \ 2\ )\}$ be a set of parameters, where S_3 is the group of permutations of $\{1,2,3\}$. Let $A = \{\epsilon_0, \epsilon_1\}$ be a subset of E and $f : A \to Z_3$ be defined by $f(\varrho) = \varrho^{-1}$ for all $\varrho \in A$. Let λ_A be an FBS set over S which is defined in terms of its fuzzy approximate functions such that

$$\overset{+}{\lambda}(\epsilon_0)(x) = \begin{cases} 0.6 & \text{if } x \in \{\gamma_0,\gamma_1,\gamma_3\}, \\ 0.2 & \text{if } x \in \{\gamma_2,\gamma_4\}, \end{cases}$$

$$\overset{-}{\lambda}(\epsilon_0)(x) = \begin{cases} 0.4 & \text{if } x \in \{\gamma_0,\gamma_1,\gamma_3\}, \\ 0.5 & \text{if } x \in \{\gamma_2,\gamma_4\}, \end{cases}$$

$$\overset{+}{\lambda}(\epsilon_1)(x) = \begin{cases} 0.5 & \text{if } x \in \{\gamma_0,\gamma_1,\gamma_3\}, \\ 0.3 & \text{if } x \in \{\gamma_2,\gamma_4\}, \end{cases}$$

$$\overset{-}{\lambda}(\epsilon_2)(x) = \begin{cases} 0.4 & \text{if } x \in \{\gamma_0,\gamma_1,\gamma_3\}, \\ 0.6 & \text{if } x \in \{\gamma_2,\gamma_4\}. \end{cases}$$

The (r,t)-level subsets $\overset{(r,t)}{\lambda_A}(\epsilon_0)$ and $\overset{(r,t)}{\lambda_A}(\epsilon_1)$ of λ_A are defined as follows:

$$\overset{(r,t)}{\lambda_A}(\epsilon_0) = \begin{cases} S & \text{if } r \in (0, 0.2], \\ \{\gamma_0,\gamma_1,\gamma_3\} & \text{if } r \in (0.2, 0.6], \\ \phi & \text{if } r \in (0.6, 1], \\ \phi & \text{if } t \in [0, 0.4), \\ \{\gamma_0,\gamma_1,\gamma_3\} & \text{if } t \in [0.4, 0.5), \\ S & \text{if } t \in [0.5, 1). \end{cases}$$

$$\overset{(r,t)}{\lambda_A}(\epsilon_1) = \begin{cases} S & \text{if } r \in (0, 0.3], \\ \{\gamma_0, \gamma_1, \gamma_3\} & \text{if } r \in (0.3, 0.5], \\ \phi & \text{if } r \in (0.5, 1], \\ \phi & \text{if } t \in [0, 0.4), \\ \{\gamma_0, \gamma_1, \gamma_3\} & \text{if } t \in [0.4, 0.6), \\ S & \text{if } t \in [0.6, 1). \end{cases}$$

Here, we see that for all $r \in (0, 1]$, $t \in [0, 1)$ and $\epsilon_0, \epsilon_1 \in A$, the (r, t)-level subsets $\overset{(r,t)}{\lambda_A}(\epsilon_0) \neq \phi$ and $\overset{(r,t)}{\lambda_A}(\epsilon_1) \neq \phi$ of λ_A are quasi-ideals of S. Then, by Theorem 1, we have λ_A is an FBS quasi-ideal over S. Further, we note that $\{\gamma_0, \gamma_1, \gamma_3\}$ is not an ideal of S. Then, by Lemma 5.8, we have λ_A is not an FBS ideal over S. Thus we conclude that Remark 1 stands valid.

Proposition 5.9. *Every FBS quasi-ideal λ_A over S is an FBS bi-ideal.*

Proof. Let λ_A be an FBS quasi-ideal over S and $\varepsilon \in A$. Suppose $x, y \in S$, then $xy \in X_{xy}$. Since $X_{xy} \neq \phi$, we have

$$\begin{aligned} \overset{+}{\lambda}(\varepsilon)(xy) &\geq [(\overset{+}{\lambda} \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\lambda})](\varepsilon)(xy) \\ &= \min[(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(xy), (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(xy)] \\ &= \min\left[\bigvee_{(p,q)\in X_{xy}} \min\{\overset{+}{\lambda}(\varepsilon)(p), \overset{+}{S}(\varepsilon)(q)\}, \bigvee_{(p,q)\in X_{xy}} \min\{\overset{+}{S}(\varepsilon)(p), \overset{+}{\lambda}(\varepsilon)(q)\}\right] \\ &\geq \min[\min\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{S}(\varepsilon)(y)\}, \min\{\overset{+}{S}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)\}] \\ &= \min[\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)]. \end{aligned}$$

Similarly, we obtain

$$\overset{-}{\lambda}(\varepsilon)(xy) \leq \max[\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(y)].$$

Thus λ_A is an FBS ordered semigroup over S. Now, let $x, y, z \in S$. Then $xyz = (xy)z = x(yz)$ and hence $(xy, z), (x, yz) \in X_{xyz}$. Since $X_{xyz} \neq \phi$, we have

$$\begin{aligned} \overset{-}{\lambda}(\varepsilon)(xyz) &\leq (\overset{-}{\lambda} \circ \overset{-}{S}) \vee (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(xyz) \\ &= \max[(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(xyz), (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(xyz)] \\ &= \max\left[\bigwedge_{(p,q)\in X_{xyz}} \max\{\overset{-}{\lambda}(\varepsilon)(p), \overset{-}{S}(\varepsilon)(q)\}, \bigwedge_{(p,q)\in X_{xyz}} \max\{\overset{-}{S}(\varepsilon)(p), \overset{-}{\lambda}(\varepsilon)(q)\}\right] \\ &\leq \max[\max\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{S}(\varepsilon)(yz)\}, \max\{\overset{-}{S}(\varepsilon)(xy), \overset{-}{\lambda}(\varepsilon)(z)\}] \\ &= \max[\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(z)]. \end{aligned}$$

Similarly, we obtain

$$\overset{+}{\lambda}(\varepsilon)(xyz) \quad \geq \quad \min[\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(z)].$$

Finally, let $x, y \in S$ such that $x \leq y$. Then, since λ_A is FBS quasi-ideal, we have $\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y)$ and $\overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y)$. Therefore, λ_A is an FBS bi-ideal over S. $\square$

The converse of Proposition 5.9 is not true in general. Thus, we have the following assertion:

Remark 2. *Every FBS bi-ideal λ_A over S is not necessarily an FBS quasi-ideal over S.*

Lemma 5.10. *(cf. [5, Theorem 4]) An FBS set λ_A over S is an FBS bi-ideal over S iff the (r,t)-level subset $\overset{(r,t)}{\lambda_A}(\varepsilon) \neq \phi$ of λ_A is a bi-ideal of S, for all $r \in (0,1],\ t \in [0,1)$ and $\varepsilon \in A$.*

To justify Remark 2, we present the following example:

Example 4. *Consider the ordered semigroup $S = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ with the multiplication "$\cdot$" and the order relation "$\leq$" given below:*

$\cdot$	γ_0	γ_1	γ_2	γ_3	γ_4
γ_0	γ_0	γ_0	γ_0	γ_0	γ_0
γ_1	γ_0	γ_1	γ_0	γ_0	γ_0
γ_2	γ_0	γ_0	γ_0	γ_0	γ_0
γ_3	γ_0	γ_0	γ_0	γ_4	γ_2
γ_4	γ_0	γ_0	γ_0	γ_2	γ_0

$$\leq = \{(\gamma_0,\gamma_0), (\gamma_0,\gamma_2), (\gamma_0,\gamma_3), (\gamma_0,\gamma_4), (\gamma_1,\gamma_1), (\gamma_2,\gamma_2), (\gamma_3,\gamma_3), (\gamma_4,\gamma_4)\}.$$

Let $E = G = \{1, -1, i, -i\}$ be a set of parameters which is the multiplicative group of the fourth roots of unity. Let $A = \{1, i\} \subset E$ and $f : A \to E$ be given by $f(\varepsilon) = \varepsilon^{-1}$ for all $\varepsilon \in A$. Now, let λ_A be an FBS set over S which is defined in terms of its fuzzy approximate functions such that

$$\overset{+}{\lambda}(1)(x) = \begin{cases} 0.5 & \text{if } x \in \{\gamma_0, \gamma_1, \gamma_4\}, \\ 0.2 & \text{if } x \in \{\gamma_2, \gamma_3\}, \end{cases}$$

$$\overset{-}{\lambda}(1)(x) = \begin{cases} 0.4 & \text{if } x \in \{\gamma_0, \gamma_1, \gamma_4\}, \\ 0.6 & \text{if } x \in \{\gamma_2, \gamma_3\}, \end{cases}$$

$$\overset{+}{\lambda}(i)(x) = \begin{cases} 0.6 & \text{if } x \in \{\gamma_0, \gamma_1, \gamma_4\}, \\ 0.3 & \text{if } x \in \{\gamma_2, \gamma_3\}, \end{cases}$$

$$\overset{-}{\lambda}(-i)(x) = \begin{cases} 0.3 & \text{if } x \in \{\gamma_0, \gamma_1, \gamma_4\}, \\ 0.5 & \text{if } x \in \{\gamma_2, \gamma_3\}. \end{cases}$$

Finally, the (r,t)-level subsets $\overset{(r,t)}{\lambda_A}(1)$ and $\overset{(r,t)}{\lambda_A}(i)$ of λ_A are defined as follows:

$$\overset{(r,t)}{\lambda_A}(1) = \begin{cases} S & \text{if } r \in (0, 0.2], \\ \{\gamma_0, \gamma_1, \gamma_4\} & \text{if } r \in (0.2, 0.5], \\ \phi & \text{if } r \in (0.5, 1], \\ \phi & \text{if } t \in [0, 0.4), \\ \{\gamma_0, \gamma_1, \gamma_4\} & \text{if } t \in [0.4, 0.6), \\ S & \text{if } t \in [0.6, 1), \end{cases}$$

$$\overset{(r,t)}{\lambda_A}(i) = \begin{cases} S & \text{if } r \in (0, 0.3], \\ \{\gamma_0, \gamma_1, \gamma_4\} & \text{if } r \in (0.3, 0.6], \\ \phi & \text{if } r \in (0.6, 1], \\ \phi & \text{if } t \in [0, 0.3), \\ \{\gamma_0, \gamma_1, \gamma_4\} & \text{if } t \in [0.3, 0.5), \\ S & \text{if } t \in [0.5, 1). \end{cases}$$

Here, we note that for all $r \in (0, 1]$, $t \in [0, 1)$ and $i, 1 \in A$, the (r,t)-level subsets $\overset{(r,t)}{\lambda_A}(1) \neq \phi$ and $\overset{(r,t)}{\lambda_A}(i) \neq \phi$ of λ_A are bi-ideals of S. Then, by Lemma 5.10, we have λ_A is an FBS bi-ideal over S. Further, we note that $\{\gamma_0, \gamma_1, \gamma_4\}$ is not a quasi-ideal of S. Then, by Theorem 1, we have λ_A is not an FBS quasi-ideal over S. Thus, we conclude that Remark 2 stands valid.

In the following proposition, the condition of regularity is imposed on S under which an FBS bi-ideal λ_A over S happens to be an FBS quasi-ideal.

Proposition 5.11. *Let S be regular. Then every FBS bi-ideal over S is an FBS quasi-ideal.*

Proof. It is straightforward. □

By virtue of Propositions 5.9 and 5.11, we formulate the following result:

Proposition 5.12. *Let S be regular. Then the concepts of FBS bi-ideal and FBS quasi-ideal over S coincide.*

Proposition 5.13. *Let λ_A be an FBS set over S. Then, for all $\varepsilon \in A$ and $x, y \in S$, the following axioms are satisfied:*

(1) $(\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(xy) \geq \overset{+}{\lambda}(\varepsilon)(y)$.

(2) $(\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(xy) \leq \overset{-}{\lambda}(\varepsilon)(y)$.

(3) $(\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(xy) \geq (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(y)$.

(4) $(\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(xy) \leq (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(y)$.

Proof. It is straightforward. □

Proposition 5.14. *Let λ_A be an FBS set over S. Then, for all $\varepsilon \in A$ and $x, y \in S$, the following assertions hold:*

(1) $(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(xy) \geq \overset{+}{\lambda}(\varepsilon)(x)$.
(2) $(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(xy) \leq \overset{-}{\lambda}(\varepsilon)(x)$.
(3) $(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(xy) \geq (\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(x)$.
(4) $(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(xy) \leq (\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(x)$.

Proof. It is straightforward. □

Proposition 5.15. *Let λ_A be an FBS set over S and $x \leq y$ for any $x, y \in S$. Then, for all $\varepsilon \in A$, the following axioms are satisfied:*

(i) $(\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(x) \geq (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(y)$.
(ii) $(\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(x) \leq (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(y)$.

Proof. It is straightforward. □

Proposition 5.16. *Let λ_A be an FBS set over S and $x \leq y$ for any $x, y \in S$. Then, for all $\varepsilon \in A$, the following axioms are satisfied:*

(i) $(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(x) \geq (\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(y)$.
(ii) $(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(x) \leq (\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(y)$.

Proof. It is straightforward. □

Proposition 5.17. *Let λ_A be an FBS set over S. Then, for all $\varepsilon \in A$ and $x, y \in S$, the following assertions are satisfied:*

(i) $(\overset{+}{\lambda} \vee (\overset{+}{S} \circ \overset{+}{\lambda}))(\varepsilon)(xy) \geq (\overset{+}{\lambda} \vee (\overset{+}{S} \circ \overset{+}{\lambda}))(\varepsilon)(y)$.
(ii) $(\overset{-}{\lambda} \wedge (\overset{-}{S} \circ \overset{-}{\lambda}))(\varepsilon)(xy) \leq (\overset{-}{\lambda} \wedge (\overset{-}{S} \circ \overset{-}{\lambda}))(\varepsilon)(y)$.

Proof. It is straightforward. □

Proposition 5.18. *Let λ_A be an FBS set over S. Then, for all $\varepsilon \in A$ and $x, y \in S$, the following assertions are satisfied:*

(i) $(\overset{+}{\lambda} \vee (\overset{+}{\lambda} \circ \overset{+}{S}))(\varepsilon)(xy) \geq (\overset{+}{\lambda} \vee (\overset{+}{\lambda} \circ \overset{+}{S}))(\varepsilon)(x)$.
(ii) $(\overset{-}{\lambda} \wedge (\overset{-}{\lambda} \circ \overset{-}{S}))(\varepsilon)(xy) \leq (\overset{-}{\lambda} \wedge (\overset{-}{\lambda} \circ \overset{-}{S}))(\varepsilon)(x)$.

Proof. It is straightforward. □

Proposition 5.19. *Let λ_A be an FBS set over S. If, for all $x, y \in S$ such that $x \leq y$, we have*

$$\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y),$$

for all $\varepsilon \in A$, then $\lambda_A \,\tilde{\cup}\, (S_A \circ \lambda_A)$ is an FBS left ideal over S.

Proof. Let $\varepsilon \in A$ and $x, y \in S$. By Proposition 5.17**,** we have

$$(\overset{+}{\lambda} \vee (\overset{+}{S} \circ \overset{+}{\lambda}))(\varepsilon)(xy) \quad \geq \quad (\overset{+}{\lambda} \vee (\overset{+}{S} \circ \overset{+}{\lambda}))(\varepsilon)(y)$$

and

$$(\overset{-}{\lambda} \wedge (\overset{-}{S} \circ \overset{-}{\lambda}))(\varepsilon)(xy) \quad \leq \quad (\overset{-}{\lambda} \wedge (\overset{-}{S} \circ \overset{-}{\lambda}))(\varepsilon)(y).$$

Now, let $x \leq y$. Then, by Proposition 5.15, we have

$$(1) \qquad (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(x) \quad \geq \quad (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(y),$$

$$(2) \qquad (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(x) \quad \leq \quad (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(y).$$

Further, by the hypothesis, we have

$$(3) \qquad \overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y).$$

Then, using (1), (2) and (3), we have

$$\begin{aligned}(\overset{+}{\lambda} \vee (\overset{+}{S} \circ \overset{+}{\lambda}))(\varepsilon)(x) &= \max\{\overset{+}{\lambda}(\varepsilon)(x), (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(x)\} \\ &\geq \max\{\overset{+}{\lambda}(\varepsilon)(y), (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(y)\} \\ &= (\overset{+}{\lambda} \vee (\overset{+}{S} \circ \overset{+}{\lambda}))(\varepsilon)(y)\end{aligned}$$

and

$$\begin{aligned}(\overset{-}{\lambda} \wedge (\overset{-}{S} \circ \overset{-}{\lambda}))(\varepsilon)(x) &= \min\{\overset{-}{\lambda}(\varepsilon)(x), (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(x)\} \\ &\leq \min\{\overset{-}{\lambda}(\varepsilon)(y), (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(y)\} \\ &= (\overset{-}{\lambda} \wedge (\overset{-}{S} \circ \overset{-}{\lambda}))(\varepsilon)(y).\end{aligned}$$

This completes the proof. □

Similarly, one can prove the following proposition:

Proposition 5.20. *Let λ_A be an FBS set over S. If, for all $x, y \in S$ such that $x \leq y$, we have*

$$\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y),$$

for all $\varepsilon \in A$, then $\lambda_A \,\widetilde{\cup}\, (\lambda_A \circ S_A)$ is an FBS right ideal over S.

As an immediate consequence of Proposition 5.5 and 5.20, we establish the following result:

Proposition 5.21. *Let λ_A be an FBS set over S. If, for all $x, y \in S$ such that $x \leq y$, we have*

$$\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y)$$

for all $\varepsilon \in A$, then $\lambda_A \,\widetilde{\cup}\, (\lambda_A \circ S_A)$ is an FBS quasi-ideal over S.

Similarly, as an immediate consequence of Proposition 5.6 and 5.19, we formulate the following result:

Proposition 5.22. *Let λ_A be an FBS set over S. If, for all $x, y \in S$ such that $x \leq y$, we have*

$$\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y)$$

for all $\varepsilon \in A$, then $\lambda_A \mathbin{\widetilde{\cup}} (S_A \circ \lambda_A)$ is an FBS quasi-ideal ideal over S.

In the light of Propositions 5.9 and 5.21, we formulate the following result:

Corollary 1. *Let λ_A be an FBS set over S. If, for all $x, y \in S$ such that $x \leq y$, we have*

$$\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y),$$

for all $\varepsilon \in A$, then $\lambda_A \mathbin{\widetilde{\cup}} (\lambda_A \circ S_A)$ is an FBS bi-ideal over S.

In the same fashion, combining Propositions 5.9 and 5.22, we establish the following corollary:

Corollary 2. *Let λ_A be an FBS set over S. If, for all $x, y \in S$ such that $x \leq y$, we have*

$$\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y),$$

for all $\varepsilon \in A$, then $\lambda_A \mathbin{\widetilde{\cup}} (S_A \circ \lambda_A)$ is an FBS bi-ideal ideal over S.

To prove Theorem 3, we need the following result:

Lemma 5.23. *(cf. [3, Proposition 3.5]) Let λ_A, τ_A and δ_A be FBS sets over S. Then, the following distributive laws hold:*

(i) $\lambda_A \mathbin{\widetilde{\cap}} (\tau_A \mathbin{\widetilde{\cup}} \delta_A) = (\lambda_A \mathbin{\widetilde{\cap}} \tau_A) \mathbin{\widetilde{\cup}} (\lambda_A \mathbin{\widetilde{\cap}} \delta_A)$.
(ii) $\lambda_A \mathbin{\widetilde{\cup}} (\tau_A \mathbin{\widetilde{\cap}} \delta_A) = (\lambda_A \mathbin{\widetilde{\cup}} \tau_A) \mathbin{\widetilde{\cap}} (\lambda_A \mathbin{\widetilde{\cup}} \delta_A)$.

The following theorem characterizes FBS quasi-ideals over S by FBS left and FBS right ideals over S.

Theorem 3. *Let λ_A be an FBS set over S. Then the following conditions are equivalent.*

(i) *λ_A is an FBS quasi-ideal over S.*
(ii) *There exist an FBS right ideal γ_A and an FBS left ideal δ_A over S such that $\lambda_A = \gamma_A \mathbin{\widetilde{\cap}} \delta_A$.*

Proof. Let λ_A be an FBS quasi-ideal over S. By Proposition 5.5 and 5.6, we have $\lambda_A \mathbin{\widetilde{\cup}} (\lambda_A \circ S_A)$ and $\lambda_A \mathbin{\widetilde{\cup}} (S_A \circ \lambda_A)$ are respectively FBS right and FBS left ideals over S. Then, by Lemma 5.23, we have

$$\begin{aligned}(\lambda_A \,\widetilde{\cup}\, (S_A \circ \lambda_A)) \,\widetilde{\cap}\, (\lambda_A \,\widetilde{\cup}\, (\lambda_A \circ S_A)) &= (\lambda_A \,\widetilde{\cup}\, (S_A \circ \lambda_A)) \,\widetilde{\cap}\, \lambda_A) \,\widetilde{\cup} \\ &\quad ((\lambda_A \,\widetilde{\cup}\, (S_A \circ \lambda_A)) \,\widetilde{\cap}\, (\lambda_A \circ S_A)) \\ &= (\lambda_A \,\widetilde{\cap}\, \lambda_A) \,\widetilde{\cup}\, ((S_A \circ \lambda_A) \,\widetilde{\cap}\, \lambda_A) \,\widetilde{\cup} \\ &\quad (\lambda_A \cap (\lambda_A \circ S_A)) \,\widetilde{\cup}\, ((S_A \circ \lambda_A) \,\widetilde{\cap}\, (\lambda_A \circ S_A)) \\ &= \lambda_A \,\widetilde{\cup}\, ((S_A \circ \lambda_A) \,\widetilde{\cap}\, \lambda_A) \,\widetilde{\cup}\, (\lambda_A \,\widetilde{\cap}\, (\lambda_A \circ S_A)) \\ &\quad \widetilde{\cup}((S_A \circ \lambda_A) \,\widetilde{\cap}\, (\lambda_A \circ S_A)). \qquad (1)\end{aligned}$$

Since λ_A is an FBS quasi-ideal over S, we have

$$(2) \qquad (\lambda_A \circ S_A) \,\widetilde{\cap}\, (S_A \circ \lambda_A) \quad \widetilde{\preceq} \quad \lambda_A.$$

Moreover,

$$(3) \qquad (S_A \circ \lambda_A) \,\widetilde{\cap}\, \lambda_A \,\widetilde{\preceq}\, \lambda_A$$

and

$$(4) \qquad \lambda_A \,\widetilde{\cap}\, (\lambda_A \circ S_A) \,\widetilde{\preceq}\, \lambda_A.$$

Hence, applying (2), (3) and (4) to (1), we obtain

$$(\lambda_A \,\widetilde{\cup}\, (S_A \circ \lambda_A)) \,\widetilde{\cap}\, (\lambda_A \,\widetilde{\cup}\, (\lambda_A \circ S_A)) \quad = \quad \lambda_A.$$

Thus Condition (ii) holds.
Conversely, assume that (ii) holds. If $X_x = \phi$, then

$$(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(x) \quad = \quad 0 = (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(x)$$

and

$$(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(x) \quad = \quad 1 = (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(x).$$

This implies that

$$\begin{aligned}((\overset{+}{\lambda} \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\lambda}))(\varepsilon)(x) &= \min\{(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(x), (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(x)\} \\ &= 0 \leq \overset{+}{\lambda}(\varepsilon)(x)\end{aligned}$$

and

$$\begin{aligned}((\overset{-}{\lambda} \circ \overset{-}{S}) \vee (\overset{-}{S} \circ \overset{-}{\lambda}))(\varepsilon)(x) &= \max\{(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(x), (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(x)\} \\ &= 1 \geq \overset{-}{\lambda}(\varepsilon)(x).\end{aligned}$$

So, in this case, we obtain

$$(\overset{+}{\lambda} \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\lambda}) \quad \leq \quad \overset{+}{\lambda}, \quad (\overset{-}{\lambda} \circ \overset{-}{S}) \vee (\overset{-}{S} \circ \overset{-}{\lambda}) \geq \overset{-}{\lambda}.$$

This means that

$$(\lambda_A \circ S_A) \,\widetilde{\cap}\, (S_A \circ \lambda_A) \quad \widetilde{\preceq} \quad \lambda_A.$$

If $X_x \neq \phi$, then

$$\begin{aligned} (\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(x) &= \bigvee_{(y,z)\in X_x} \min\{(\overset{+}{\lambda}(\varepsilon)(y), \overset{+}{S}(\varepsilon)(z)\} \\ &= \bigvee_{(y,z)\in X_x} \{\overset{+}{\lambda}(\varepsilon)(y)\} \qquad (5) \end{aligned}$$

and

$$\begin{aligned} (\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(x) &= \bigwedge_{(y,z)\in X_x} \max\{(\overset{-}{\lambda}(\varepsilon)(y), \overset{-}{S}(\varepsilon)(z)\} \\ &= \bigwedge_{(y,z)\in X_x} \{\overset{-}{\lambda}(\varepsilon)(y)\}. \qquad (6) \end{aligned}$$

Similarly, we have

$$\begin{aligned} (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(x) &= \bigvee_{(y,z)\in X_x} \min\{(\overset{+}{S}(\varepsilon)(y), \overset{+}{\lambda}(\varepsilon)(z)\} \\ &= \bigvee_{(y,z)\in X_x} \{\overset{+}{\lambda}(\varepsilon)(z)\} \qquad (7) \end{aligned}$$

and

$$\begin{aligned} (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(x) &= \bigwedge_{(y,z)\in X_x} \max\{(\overset{-}{S}(\varepsilon)(y), \overset{-}{\lambda}(\varepsilon)(z)\} \\ &= \bigwedge_{(y,z)\in X_x} \{\overset{-}{\lambda}(\varepsilon)(z)\}. \qquad (8) \end{aligned}$$

Further, for $(y, z) \in X_x$, we have $x \leq yz$. Then, since γ_A is an FBS right ideal over S, we have

$$(9) \qquad \overset{+}{\gamma}(\varepsilon)(x) \geq \overset{+}{\gamma}(\varepsilon)(yz) \geq \overset{+}{\gamma}(\varepsilon)(y) \geq \overset{+}{\lambda}(\varepsilon)(y)$$

and

$$(10) \qquad \overset{-}{\gamma}(\varepsilon)(x) \leq \overset{-}{\gamma}(\varepsilon)(yz) \leq \overset{-}{\gamma}(\varepsilon)(y) \leq \overset{-}{\lambda}(\varepsilon)(y).$$

Similarly, since δ_A is an FBS left ideal over S, we have

$$(11) \qquad \overset{+}{\delta}(\varepsilon)(x) \geq \overset{+}{\delta}(\varepsilon)(yz) \geq \overset{+}{\delta}(\varepsilon)(z) \geq \overset{+}{\lambda}(\varepsilon)(z)$$

and

$$(12) \qquad \overset{-}{\delta}(\varepsilon)(x) \leq \overset{-}{\delta}(\varepsilon)(yz) \leq \overset{-}{\delta}(\varepsilon)(z) \leq \overset{-}{\lambda}(\varepsilon)(z).$$

Thus, applying (9) to (5) and (10) to (6), we obtain

$$(\overset{+}{\lambda} \circ \overset{+}{S})(x) = \bigvee_{(y,z)\in X_x} \{\overset{+}{\lambda}(\varepsilon)(y)\} \leq \overset{+}{\gamma}(\varepsilon)(x),$$

$$(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(x) \quad = \quad \bigvee_{(y,z)\in X_x} \{\overset{-}{\lambda}(\varepsilon)(y)\} \geq \overset{-}{\gamma}(\varepsilon)(x).$$

Similarly, applying (11) to (7) and (12) to (8), we obtain

$$(\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(x) = \bigvee_{(y,z)\in X_x} \{\overset{+}{\lambda}(\varepsilon)(z)\} \leq \overset{+}{\delta}(\varepsilon)(x),$$

$$(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(x) = \bigvee_{(y,z)\in X_x} \{\overset{-}{\lambda}(\varepsilon)(z)\} \geq \overset{-}{\delta}(\varepsilon)(x).$$

Thus, we have

$$\begin{aligned}((\overset{+}{\lambda} \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\lambda}))(\varepsilon)(x) \quad &= \quad \min\{(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(x), (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(x)\} \\ &\leq \quad \min\{\overset{+}{\gamma}(\varepsilon)(x), \overset{+}{\delta}(\varepsilon)(x)\} = \overset{+}{\lambda}(\varepsilon)(x)\end{aligned}$$

and

$$\begin{aligned}((\overset{-}{\lambda} \circ \overset{-}{S}) \vee (\overset{-}{S} \circ \overset{-}{\lambda}))(\varepsilon)(x) \quad &= \quad \max\{(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(x), (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(x)\} \\ &\geq \quad \max\{\overset{-}{\gamma}(\varepsilon)(x), \overset{-}{\delta}(\varepsilon)(x)\} = \overset{-}{\lambda}(\varepsilon)(x).\end{aligned}$$

Then, it follows that

$$(\overset{+}{\lambda} \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\lambda}) \quad \leq \quad \overset{+}{\lambda}, \quad (\overset{-}{\lambda} \circ \overset{-}{S}) \vee (\overset{-}{S} \circ \overset{-}{\lambda}) \geq \overset{-}{\lambda}.$$

Thus, we have

$$(\lambda_A \circ S_A) \,\widetilde{\cap}\, (S_A \circ \lambda_A) \quad \widetilde{\preceq} \quad \lambda_A.$$

This completes the proof. □

6. Characterizations of weakly-regular ordered semigroups in terms of their fuzzy bipolar soft quasi-ideals

In this section, we give the main theorem that characterizes weakly-regular ordered semigroups by means of their FBS quasi-ideals.

An ordered semigroup $(S, \cdot, \leq)$ is right weakly (resp., left weakly) regular iff, for every $a \in S$, there exist $x, y \in S$ such that $a \leq axay$ (resp., $a \leq xaya$). Equivalently, S is right weakly (resp, left weakly) regular iff $a \in (aSaS]$(resp,. $a \in (SaSa]$), for every $a \in S$. Moreover, S is weakly regular if it is both left weakly-regular and right weakly-regular [34].

Definition 15. *An FBS set λ_A over S is called idempotent iff $\lambda_A \circ \lambda_A = \lambda_A$ iff $\overset{+}{\lambda} \circ \overset{+}{\lambda} = \overset{+}{\lambda}$ and $\overset{-}{\lambda} \circ \overset{-}{\lambda} = \overset{-}{\lambda}$.*

To prove Theorem 4, we need the follpowing lemmas:

Lemma 6.1. *(cf. [3, Proposition 5.12]) Let Γ_A be an FBS set over S. Then $S_A \circ \Gamma_A$ is an FBS left ideal over S.*

Lemma 6.2. *(cf. [3, Proposition 5.13]) Let Γ_A be an FBS set over S. Then $\Gamma_A \circ S_A$ is an FBS right ideal over S.*

Lemma 6.3. *The following assertions are equivalent on S:*

(i) *S is left weakly-regular.*
(ii) *$\lambda_A = \lambda_A \circ \lambda_A$, for every FBS left ideal over S.*

Proof. It is straightforward. □

Lemma 6.4. *The following assertions are equivalent on S.*

(i) *S is right weakly-regular.*
(ii) *$\lambda_A = \lambda_A \circ \lambda_A$, for every FBS right ideal over S.*

Proof. It is straightforward. □

In the following, we establish the main theorem that characterizes weakly-regular ordered semigroups by their FBS quasi-ideals.

Theorem 4. *The following assertions are equivalent on S.*

(i) *S is weakly-regular.*
(ii) *For every FBS quasi-ideal λ_A over S,*

$$\lambda_A = (\lambda_A \circ S_A)^2 \,\tilde{\cap}\, (S_A \circ \lambda_A)^2.$$

Proof. First assume that S is weakly regular. Let λ_A be an FBS quasi-ideal over S. By Lemmas 6.1 and 6.2, we have $S_A \circ \lambda_A$ and $\lambda_A \circ S_A$ are respectively FBS left and FBS right ideals over S. By Lemmas 6.3 and 6.4, we have $S_A \circ \lambda_A$ and $\lambda_A \circ S_A$ are respectively idempotent because S is left weakly-regular and right weakly-regular, being weakly-regular. Thus, we have

$$\begin{aligned}(\lambda_A \circ S_A)^2 \,\tilde{\cap}\, (S_A \circ \lambda_A)^2 &= (\lambda_A \circ S_A) \,\tilde{\cap}\, (S_A \circ \lambda_A) \\ &\tilde{\preceq} \lambda_A. \qquad (1)\end{aligned}$$

Now, let us prove the reverse inclusion. For this, let $\varepsilon \in A$ and $a \in S$. Then, since S is left weakly-regular, there exist $x, y \in S$ such that $a \leq axay = (ax)(ay)$. Similarly, since S is right weakly-regular, there exist $x, y \in S$ such that $a \leq xaya = (xa)(ya)$. Thus $(ax, ay) \in X_a$ and $(xa, ya) \in X_a$. Since $X_a \neq \phi$, we have

$$\begin{aligned}(\bar{\lambda} \circ \bar{S})^2(\varepsilon)(a) &= \bigwedge_{(p,q)\in X_a} \max[(\bar{\lambda} \circ \bar{S})(\varepsilon)(p), (\bar{\lambda} \circ \bar{S})(\varepsilon)(q)] \\ &\leq \max[(\bar{\lambda} \circ \bar{S})(\varepsilon)(ax), (\bar{\lambda} \circ \bar{S})(\varepsilon)(ay)] \\ &= \max[\bigwedge_{(r,s)\in X_{ax}} \max\{\bar{\lambda}(\varepsilon)(r), \bar{S}(\varepsilon)(s)\}, \bigwedge_{(u,v)\in X_{ay}} \max\{\bar{\lambda}(\varepsilon)(u), \bar{S}(\varepsilon)(v)\}] \\ &\leq \max[\max\{\bar{\lambda}(\varepsilon)(a), \bar{S}(\varepsilon)(x)\}, \max\{\bar{\lambda}(\varepsilon)(a), \bar{S}(\varepsilon)(y)\}] \\ &= \max[\bar{\lambda}(\varepsilon)(a), \bar{\lambda}(\varepsilon)(a)] = \bar{\lambda}(\varepsilon)(a).\end{aligned}$$

Similarly,

$$(\overset{+}{\lambda} \circ \overset{+}{S})^2(\varepsilon)(a) \quad \geq \quad \min[\overset{+}{\lambda}(\varepsilon)(a), \overset{+}{\lambda}(\varepsilon)(a)] = \overset{+}{\lambda}(\varepsilon)(a).$$

Thus, it follows that

$$(2) \qquad \overset{+}{\lambda} \leq (\overset{+}{\lambda} \circ \overset{+}{S})^2, \quad \overset{-}{\lambda} \geq (\overset{-}{\lambda} \circ \overset{-}{S})^2.$$

Moreover, we have

$$\begin{aligned}
(\overset{+}{S} \circ \overset{+}{\lambda})^2(\varepsilon)(a) &= \bigvee_{(p,q)\in X_a} \min[(\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(p), (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(q)] \\
&\geq \min[(\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(xa), (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(ya)] \\
&= \min[\bigvee_{(u,v)\in X_{xa}} \min\{\overset{+}{S}(\varepsilon)(r), \overset{+}{\lambda}(\varepsilon)(s)\}, \bigvee_{(r,s)\in X_{ya}} \min\{\overset{+}{S}(\varepsilon)(u), \overset{+}{\lambda}(\varepsilon)(v)\}] \\
&\geq \min[\min\{\overset{+}{S}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(a)\}, \min\{\overset{+}{S}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(a)\}] \\
&= \min[\overset{+}{\lambda}(\varepsilon)(a), \overset{+}{\lambda}(\varepsilon)(a)] = \overset{+}{\lambda}(\varepsilon)(a).
\end{aligned}$$

Similarly, we obtain

$$(\overset{-}{S} \circ \overset{-}{\lambda})^2(\varepsilon)(a) \quad \leq \quad \overset{-}{\lambda}(\varepsilon)(a).$$

Thus

$$(3) \qquad \overset{+}{\lambda} \leq (\overset{+}{S} \circ \overset{+}{\lambda})^2, \quad \overset{-}{\lambda} \geq (\overset{-}{S} \circ \overset{-}{\lambda})^2.$$

Then, from (2) and (3), we obtain

$$\overset{+}{\lambda} \quad \leq \quad (\overset{+}{\lambda} \circ \overset{+}{S})^2 \wedge (\overset{+}{S} \circ \overset{+}{\lambda})^2$$

and

$$\overset{-}{\lambda} \quad \geq \quad (\overset{-}{\lambda} \circ \overset{-}{S})^2 \vee (\overset{-}{S} \circ \overset{-}{\lambda})^2.$$

This means that

$$(4) \qquad \lambda_A \quad \tilde{\preceq} \quad (\lambda_A \circ S_A)^2 \,\tilde{\cap}\, (S_A \circ \lambda_A)^2.$$

Therefore, by (1) and (4), we have Assertion (ii) is satisfied.

Conversely, let λ_A be an FBS right ideal over S. By Proposition 5.5, we have λ_A is an FBS quasi-ideal over S. Then, we have

$$\overset{+}{\lambda} \quad = \quad (\overset{+}{\lambda} \circ \overset{+}{S})^2 \wedge (\overset{+}{S} \circ \overset{+}{\lambda})^2 \leq (\overset{+}{\lambda} \circ \overset{+}{S})^2 \leq \overset{+}{\lambda} \circ \overset{+}{\lambda} \leq \overset{+}{\lambda}.$$

Similarly,

$$\overset{-}{\lambda} \quad = \quad (\overset{-}{\lambda} \circ \overset{-}{S})^2 \vee (\overset{-}{S} \circ \overset{-}{\lambda})^2 \geq (\overset{-}{\lambda} \circ \overset{-}{S})^2 \geq \overset{-}{\lambda} \circ \overset{-}{\lambda} \geq \overset{-}{\lambda}.$$

Thus, we obtain $\lambda_A = \lambda_A \circ \lambda_A$. Then, by Lemma 6.4, we have S is right weakly-regular. Similarly, we prove that S is left weakly-regular. Therefore, S is a weakly-regular ordered semigroup. $\square$

7. CHARACTERIZATION OF INTRA-REGULAR AND LEFT WEAKLY-REGULAR ORDERED SEMIGROUPS IN TERMS OF THEIR FUZZY BIPOLAR SOFT LEFT, FUZZY BIPOLAR SOFT RIGHT AND FUZZY BIPOLAR SOFT QUASI-IDEALS

In this section, we give a characterization of ordered semigroups (having the identity element 1) that are both intra-regular and left weakly-regular by their FBS left, FBS right and FBS quasi-ideals.

To begin, let us recall the following:

An ordered semigroup S is said to be intra-regular iff, for every $a \in S$, there exist $x, y \in S$ such that $a \leq xa^2y$ [10].

Lemma 7.1. *[4] Let $\mathcal{P}$ be a non-empty subset of S. The following assertions are equivalent on S:*

(1) *$\mathcal{P}$ is a right (resp., left, two-sided) ideal of S.*
(2) *$\overset{\mathcal{P}}{\chi}_A$ is an FBS right (resp., left, two-sided) ideal over S.*

The following characterization of the ordered semigroups (with the identity element 1) that are both intra-regular and left weakly-regular is due to Shabir and Khan [34].

Lemma 7.2. *Let S contains the identity element 1. Then the following assertions are equivalent on S:*

(1) *S is both intra-regular and left weakly-regular.*
(2) *$L \cap R \cap Q \subseteq (LRQ]$ for every quasi-ideal Q, every left ideal L and every right ideal R of S.*
(3) *$L(a) \cap R(a) \cap Q(a) \subseteq (L(a)R(a)Q(a)]$ for every $a \in S$.*

In the following, we give another characterization of the ordered semigroups (with the identity element 1), which are both intra-regular and left weakly-regular, by means of their FBS left, FBS right and FBS quasi-ideals.

Theorem 5. *Let S contains the identity element 1. Then the following axioms are equivalent on S.*

(i) *S is both intra-regular and left weakly-regular.*
(ii) *For every FBS left ideal λ_A, every FBS right ideal γ_A and every FBS quasi-ideal δ_A over S,*

$$\lambda_A \,\widetilde{\cap}\, \gamma_A \,\widetilde{\cap}\, \delta_A \quad \widetilde{\preceq} \quad \lambda_A \circ \gamma_A \circ \delta_A.$$

Proof. First assume that Axiom (i) holds. Let λ_A be an FBS left ideal, γ_A be an FBS right ideal and δ_A be an FBS quasi-ideal over S. Let $a \in S$. Then there exist $x, y \in S$ such that $a \leq xa^2y$ because S is intra-regular. Moreover, since S is left-weakly regular, there exist $u, v \in S$ such that $a \leq uava$. Thus, we have

$$a \leq uava \leq u(xa^2y)va = ((ux)a)(a(yv)a).$$

Then $((ux)a, a(yv)a) \in X_a$. Since $X_a \neq \phi$, thus, for all $\varepsilon \in A$, we have

$$\begin{aligned}
(\bar{\lambda} \circ \bar{\gamma} \circ \bar{\delta})(\varepsilon)(a) &= \bigwedge_{(p,q)\in X_a} \max[(\bar{\lambda}(\varepsilon)(p), (\bar{\gamma} \circ \bar{\delta})(\varepsilon)(q)] \\
&\leq \max[\bar{\lambda}(\varepsilon)((ux)a), (\bar{\gamma} \circ \bar{\delta})(\varepsilon)(a(yv)a)] \\
&= \max\left\{\bar{\lambda}(\varepsilon)((ux)a), \bigwedge_{(p_1,q_1)\in X_{(a(yv)a)}} \max\{\bar{\gamma}(\varepsilon)(p_1), \bar{\delta}(\varepsilon)(q_1)\right\} \\
&\leq \max\{\bar{\lambda}(\varepsilon)((ux)a), \max\{\bar{\gamma}(\varepsilon)(a(yv)), \bar{\delta}(\varepsilon)(a)\} \\
&= \max[\bar{\lambda}(\varepsilon)((ux)a), \bar{\gamma}(\varepsilon)(a(yv)), \bar{\delta}(\varepsilon)(a)] \\
&= \max[\bar{\lambda}(\varepsilon)(a), \bar{\gamma}(\varepsilon)(a), \bar{\delta}(\varepsilon)(a)] \\
&= (\bar{\lambda} \vee \bar{\gamma} \vee \bar{\delta})(\varepsilon)(a).
\end{aligned}$$

Similarly, we obtain

$$(\overset{+}{\lambda} \circ \overset{+}{\gamma} \circ \overset{+}{\delta})(\varepsilon)(a) \quad \geq \quad (\overset{+}{\lambda} \wedge \overset{+}{\gamma} \wedge \overset{+}{\delta})(\varepsilon)(a).$$

Thus Axiom (ii) is satisfied.

Conversely, let $\varepsilon \in A$ and $a \in S$. To show that Axiom (i) holds, it is enough to prove by Lemma 7.2 that

$$L(a) \cap R(a) \cap Q(a) \quad \subseteq \quad (L(a)R(a)Q(a)] \quad \forall\, a \in S.$$

Let $b \in L(a) \cap R(a) \cap Q(a)$ where $L(a)$ is a left, $R(a)$ a right and $Q(a)$ a quasi ideal of S generated by a. Then, by Lemma 7.1, we have $\overset{L(a)}{\chi_A}$ and $\overset{R(a)}{\chi_A}$ are respectively FBS left and FBS right ideals over S and, similarly, by Theorem 2, we have $\overset{Q(a)}{\chi_A}$ is an FBS quasi-ideal over S. By the hypothesis, we have

$$\begin{aligned}
(\overset{+}{\chi}_{L(a)} \circ \overset{+}{\chi}_{R(a)} \circ \overset{+}{\chi}_{Q(a)})(\varepsilon)(b) &\geq (\overset{+}{\chi}_{L(a)} \wedge \overset{+}{\chi}_{R(a)} \wedge \overset{+}{\chi}_{Q(a)})(\varepsilon)(b) \\
&= \min\{\overset{+}{\chi}_{L(a)}(\varepsilon)(b), \overset{+}{\chi}_{R(a)}(\varepsilon)(b), \overset{+}{\chi}_{Q(a)})(\varepsilon)(b)\} \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
(\bar{\chi}_{L(a)} \circ \bar{\chi}_{R(a)} \circ \bar{\chi}_{Q(a)})(\varepsilon)(b) &\leq (\bar{\chi}_{L(a)} \vee \bar{\chi}_{R(a)} \vee \bar{\chi}_{Q(a)})(\varepsilon)(b) \\
&= \max\{\bar{\chi}_{L(a)}(\varepsilon)(b), \bar{\chi}_{R(a)}(\varepsilon)(b), \bar{\chi}_{Q(a)})(\varepsilon)(b)\} \\
&= 0.
\end{aligned}$$

This implies that

$$(\overset{+}{\chi}_{L(a)} \circ \overset{+}{\chi}_{R(a)} \circ \overset{+}{\chi}_{Q(a)})(\varepsilon)(b) \quad = \quad 1$$

and

$$(\bar{\chi}_{L(a)} \circ \bar{\chi}_{R(a)} \circ \bar{\chi}_{Q(a)})(\varepsilon)(b) \quad = \quad 0.$$

Further, by Lemma 4.3, we have

$$\overset{+}{\chi}_{L(a)} \circ \overset{+}{\chi}_{R(a)} \circ \overset{+}{\chi}_{Q(a)} \quad = \quad \overset{+}{\chi}_{(L(a)R(a)Q(a)]}$$

and

$$\bar{\chi}_{L(a)} \circ \bar{\chi}_{R(a)} \circ \bar{\chi}_{Q(a)} \quad = \quad \bar{\chi}_{(L(a)R(a)Q(a)]}.$$

Thus

$$\overset{+}{\chi}_{(L(a)R(a)Q(a)]} = 1, \quad \bar{\chi}_{(L(a)R(a)Q(a)]} = 0,$$

which implies that $b \in (L(a)R(a)Q(a)]$. This completes the proof. □

8. Characterizations of both left simple and right simple (resp., completely regular) ordered semigroups by their fuzzy bipolar soft quasi-ideals

In this section, we characterize the ordered semigroups that are both left and right simple by means of their FBS quasi-ideals. Moreover, we define FBS semiprime quasi-ideals in ordered semigroup theory and characterize completely regular ordered semigroups by their FBS (semiprime) quasi-ideals. It is proved, among others, that S is completely regular iff every FBS quasi-ideal λ_A over S is an FBS semiprime quasi-ideal.

An ordered semigroup S is said to be right (resp., left) regular iff, for every $a \in S$, there exists some $x \in S$ such that $a \leq a^2x$ (resp., $a \leq xa^2$) [7, 14]. Similarly, S is said to be completely regular iff it is regular, left regular and right regular [11].

Lemma 8.1. *[15] An ordered semigroup S is left (resp., right) simple iff $(Sa] = S$ (resp., $(aS] = S$) for every $a \in S$.*

Lemma 8.2. *Let $\sigma \in S$. Then $(\sigma S]$ (resp., $(S\sigma]$) is a quasi-ideal of S.*

Proof. Firstly, we have

$$((\sigma S]\, S] \cap (S\, (\sigma S]] \subseteq (\sigma S] \cap (S\sigma S] \subseteq (\sigma S]\,.$$

Secondly, if $x \in (\sigma S]$ and $S \ni y \leq x$, then $y \in ((\sigma S]] = (\sigma S]$. Thus, $(\sigma S]$ is a quasi-ideal of S. □

In the following, we characterize the ordered semigroups that are both left and right simple, having the identity element 1, by means of their FBS quasi-ideals.

Theorem 6. *Let S contains the identity element 1. Then the following axioms are equivalent on S:*

(1) *S is left and right simple.*
(2) *$S = (aSa]$ for all $a \in S$.*
(3) *S is regular, left and right simple.*
(4) *Every FBS quasi-ideal over S is a constant function.*

Proof. First assume that Axiom (1) holds. Let $a \in S$. It is well known by Lemma 8.1 that $S = (Sa] = (aS]$. Then, it follows that

$$S = (aS] = (a\ (Sa]] = (aSa]\,.$$

Thus Axiom (1) implies (2). Second, assume that Axiom (2) holds. Let $a \in S$. Then, by Axiom (2), we have

$$S = (aSa] \subseteq (Sa] \subseteq (S] = S$$

and

$$S = (aSa] \subseteq (aS] \subseteq (S] = S.$$

So, we obtain

$$a \in (aSa] \text{ and } S = (Sa] = (aS]\,.$$

Thus Axiom (2) implies (3). Third, we show that Axiom (3) implies (4). Let S be regular, left and right simple. Further, let λ_A be an FBS quasi-ideal over S and $a \in S$. We consider the set

$$E_S \quad = \quad \{e \in S \mid e^2 \geq e\}.$$

Since S is regular, there exists $x \in S$ such that $a \leq axa$. Then, we have

$$(ax)^2 = ax.ax = (axa)x \geq ax$$

which implies that $ax \in E_S$ and thus $E_S \neq \phi$.

(i) First, we show that λ_A is constant on E_S. Let $t \in E_S$. Since S is left and right simple, we have $(St] = S$ and $(tS] = S$. Moreover, $e \in (St]$ and $e \in (tS]$ because $e \in S$. Thus, there exist $x, y \in S$ such that $e \leq xt$ and $e \leq ty$. If $e \leq xt$, then, we have

$$e^2 = ee \leq xt \cdot xt = (xtx)t$$

which implies that $(xtx, t) \in X_{e^2}$. If $e \leq ty$, then, we have

$$e^2 = ee \leq ty \cdot ty = t(yty)$$

which implies that $(t, yty) \in X_{e^2}$. Since $X_{e^2} \neq \phi$, thus, for all $\varepsilon \in A$, we have

$$\begin{array}{rcl} \overset{+}{\lambda}(\varepsilon)(e^2) & \geq & ((\overset{+}{\lambda} \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\lambda}))(\varepsilon)(e^2) \\ & = & \min[(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(e^2), (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(e^2)] \\ & = & \min\left[\bigvee\limits_{(p,q)\in X_{e^2}} \min\{\overset{+}{\lambda}(\varepsilon)(p), \overset{+}{S}(\varepsilon)(q)\}, \bigvee\limits_{(u,v)\in X_{e^2}} \min\{\overset{+}{S}(\varepsilon)(u), \overset{+}{\lambda}(\varepsilon)(v)\}\right] \\ & \geq & \min\left[\min\{\overset{+}{\lambda}(\varepsilon)(t), \overset{+}{S}(\varepsilon)(xtx)\}, \min\{\overset{+}{S}(\varepsilon)(yty), \overset{+}{\lambda}(\varepsilon)(t)\}\right] \\ & = & \min[\overset{+}{\lambda}(\varepsilon)(t), \overset{+}{\lambda}(\varepsilon)(t)] = \overset{+}{\lambda}(\varepsilon)(t). \end{array}$$

Similarly, we obtain

$$\overset{-}{\lambda}(\varepsilon)(e^2) \quad \leq \quad \max[\overset{-}{\lambda}(\varepsilon)(t), \overset{-}{\lambda}(\varepsilon)(t)] = \overset{-}{\lambda}(\varepsilon)(t).$$

Since $e \in E_S$, we have $e^2 \geq e$. Then, for all $\varepsilon \in A$, we have $\overset{+}{\lambda}(\varepsilon)(e^2) \leq \overset{+}{\lambda}(\varepsilon)(e)$ and $\overset{-}{\lambda}(\varepsilon)(e^2) \geq \overset{-}{\lambda}(\varepsilon)(e)$, and thus we obtain

$$\overset{+}{\lambda}(\varepsilon)(e) \geq \overset{+}{\lambda}(\varepsilon)(t), \quad \overset{-}{\lambda}(\varepsilon)(e) \leq \overset{-}{\lambda}(\varepsilon)(t).$$

On the other hand, since S is left and right simple, we have $S = (Se] = (eS]$. Since $t \in S$, hence $t \in (Se]$ and $t \in (eS]$. Then there exist $u, v \in S$ such that $t \leq ue$ and $t \leq ev$. Since $t \in E_S$, then, under the same considerations as in the previous case, we have

$$\overset{+}{\lambda}(\varepsilon)(t) \geq \overset{+}{\lambda}(\varepsilon)(e), \quad \overset{-}{\lambda}(\varepsilon)(t) \leq \overset{-}{\lambda}(\varepsilon)(e).$$

Thus, we obtain

$$\overset{+}{\lambda}(\varepsilon)(t) = \overset{+}{\lambda}(\varepsilon)(e), \quad \overset{-}{\lambda}(\varepsilon)(t) = \overset{-}{\lambda}(\varepsilon)(e).$$

Therefore, λ_A is constant on E_S.

(ii) Secondly, we show that λ_A is constant on S. Let $a \in S$. Then there exists $x \in S$ such that $a \leq axa$ because S is regular. We consider the elements ax and xa in S. Thus, we have

$$(xa)^2 = x(axa) \geq xa, \quad (ax)^2 = (axa)x \geq ax,$$

which implies that $xa, ax \in E_S$. Then, by (i), we obtain

$$\overset{+}{\lambda}(\varepsilon)(xa) = \overset{+}{\lambda}(\varepsilon)(t), \quad \overset{-}{\lambda}(\varepsilon)(xa) = \overset{-}{\lambda}(\varepsilon)(t),$$

and

$$\overset{+}{\lambda}(\varepsilon)(ax) = \overset{+}{\lambda}(\varepsilon)(t), \quad \overset{-}{\lambda}(\varepsilon)(ax) = \overset{-}{\lambda}(\varepsilon)(t),$$

for all $t \in S$. Moreover, we have

$$(ax)(axa) \geq axa \geq a, \quad (axa)(xa) \geq axa \geq a,$$

which implies that $(ax, axa), (axa, xa) \in X_a$. Since $X_a \neq \phi$ and λ_A is an FBS quasi-ideal over S, then, for all $\varepsilon \in A$, we have

$$\begin{aligned}
\overset{-}{\lambda}(\varepsilon)(a) &\leq ((\overset{-}{\lambda} \circ \overset{-}{S}) \wedge (\overset{-}{S} \circ \overset{-}{\lambda}))(\varepsilon)(a) \\
&= \max[(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(a), (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(a)] \\
&= \max\left[\bigwedge_{(a,b) \in X_a} \max\{\overset{-}{\lambda}(\varepsilon)(a), \overset{-}{S}(\varepsilon)(b)\}, \bigwedge_{(c,d) \in X_a} \max\{\overset{-}{S}(\varepsilon)(c), \overset{-}{\lambda}(\varepsilon)(d)\}\right] \\
&\leq \max\left[\max\{\overset{-}{\lambda}(\varepsilon)(ax), \overset{-}{S}(\varepsilon)(axa)\}, \max\{\overset{-}{S}(\varepsilon)(axa), \overset{-}{\lambda}(\varepsilon)(xa)\}\right] \\
&= \max[\overset{-}{\lambda}(\varepsilon)(ax), \overset{-}{\lambda}(\varepsilon)(xa)] \\
&= \max[\overset{-}{\lambda}(\varepsilon)(t), \lambda(\varepsilon)(t)] = \overset{-}{\lambda}(\varepsilon)(t).
\end{aligned}$$

Similarly, we obtain that $\overset{+}{\lambda}(\varepsilon)(a) \geq \overset{+}{\lambda}(\varepsilon)(t)$. Thus $\overset{+}{\lambda}(\varepsilon)(a) \geq \overset{+}{\lambda}(\varepsilon)(t)$ and $\overset{-}{\lambda}(\varepsilon)(a) \leq \overset{-}{\lambda}(\varepsilon)(t)$. Since S is left and right simple, we have $(aS] = S$ and $(Sa] = S$. Further, since $t \in S$, there exist $u, v \in S$ such that $t \leq au$ and $t \leq va$. Then $t^2 = tt \leq a(uau)$ and $t^2 = tt \leq (vav)a$. Therefore, it follows that $(a, uau), (vav, a) \in X_{t^2}$. Since $X_{t^2} \neq \phi$, then, for all $\varepsilon \in A$, we have

$$\begin{aligned}
\overset{+}{\lambda}(\varepsilon)(t^2) &\geq ((\overset{+}{\lambda} \circ \overset{+}{S}) \wedge (\overset{+}{S} \circ \overset{+}{\lambda}))(\varepsilon)(t^2) \\
&= \min[(\overset{+}{\lambda} \circ \overset{+}{S})(\varepsilon)(t^2), (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(t^2)] \\
&= \min\left[\bigvee_{(p,q)\in X_{t^2}} \min\{\overset{+}{\lambda}(\varepsilon)(p), \overset{+}{S}(\varepsilon)(q)\}, \bigvee_{(r,s)\in X_{t^2}} \min\{\overset{+}{S}(\varepsilon)(r), \overset{+}{\lambda}(\varepsilon)(s)\}\right] \\
&\geq \min\left[\min\{\overset{+}{\lambda}(\varepsilon)(a), \overset{+}{S}(\varepsilon)(uau)\}, \min\{\overset{+}{S}(\varepsilon)(vav), \overset{+}{\lambda}(\varepsilon)(a)\}\right] \\
&= \min[\overset{+}{\lambda}(\varepsilon)(a), \overset{+}{\lambda}(\varepsilon)(a)] = \overset{+}{\lambda}(\varepsilon)(a).
\end{aligned}$$

Similarly, we obtain

$$\overset{-}{\lambda}(\varepsilon)(t^2) \quad \leq \quad \overset{-}{\lambda}(\varepsilon)(a).$$

Since $t \in E_S$, which implies that $t^2 \geq t$, we have $\overset{+}{\lambda}(\varepsilon)(t) \geq \overset{+}{\lambda}(\varepsilon)(t^2)$ and $\overset{-}{\lambda}(\varepsilon)(t) \leq \overset{-}{\lambda}(\varepsilon)(t^2)$. Then $\overset{+}{\lambda}(\varepsilon)(t) \geq \overset{+}{\lambda}(\varepsilon)(a)$ and $\overset{-}{\lambda}(\varepsilon)(t) \leq \overset{-}{\lambda}(\varepsilon)(a)$. Thus, we have

$$\overset{+}{\lambda}(\varepsilon)(a) = \overset{+}{\lambda}(\varepsilon)(t), \quad \overset{-}{\lambda}(\varepsilon)(a) = \overset{-}{\lambda}(\varepsilon)(t).$$

Finally, we show that Axiom (4) implies Axiom (1). Let $\varepsilon \in A$ and $a \in S$. Then, by Lemma 8.2, we have $(aS]$ is a quasi-ideal of S. Further, by Theorem 2, we have $\overset{(aS]}{\chi_A}$ is an FBS quasi-ideal over S. Then by the hypothesis $\overset{(aS]}{\chi_A}$ is a constant function, that is, for all $x \in S$ there exist $c_1, c_2 \in \{0, 1\} \subset [0, 1]$ such that

$$\overset{+}{\chi}_{(aS]}(\varepsilon)(x) = c_1, \quad \overset{-}{\chi}_{(aS]}(\varepsilon)(x) = c_2.$$

Let $(aS] \subset S$ and τ be an element in S such that $\tau \notin (aS]$. Then, we have $\overset{+}{\chi}_{(aS]}(\varepsilon)(\tau) = 0$ and $\overset{-}{\chi}_{(aS]}(\varepsilon)(\tau) = 1$. On the other hand, since $a^2 \in (aS]$, we have $\overset{+}{\chi}_{(aS]}(\varepsilon)(a^2) = 1$ and $\overset{-}{\chi}_{(aS]}(\varepsilon)(a^2) = 0$ which contradicts the fact that $\overset{(aS]}{\chi_A}$ is a constant function. Thus it follows that $(aS] = S$. In the same way, by symmetry, we prove that $(Sa] = S$. Therefore, S is left and right simple. $\square$

Lemma 8.3. *[11] An ordered semigroup $(S, \cdot, \leq)$ is completely regular iff $A \subseteq (A^2SA^2]$ for every $A \subseteq S$. Equivalently, S is completely regular iff $a \in (a^2Sa^2]$ for every $a \in S$.*

The following theorem characterizes completely regular ordered semigroups by their FBS quasi ideals.

Theorem 7. *The following assertions are equivalent on S:*

(i) *S is completely regular.*
(ii) *For every FBS quasi-ideal λ_A over S, we have*

$$\overset{+}{\lambda}(\varepsilon)(\sigma) = \overset{+}{\lambda}(\varepsilon)(\sigma^2), \quad \overset{-}{\lambda}(\varepsilon)(\sigma) = \overset{-}{\lambda}(\varepsilon)(\sigma^2),$$

for all $\varepsilon \in A$ and $\sigma \in S$.

Proof. First assume that Assertion (i) holds. Let λ_A be an FBS quasi-ideal over S. Let $\varepsilon \in A$ and $\sigma \in S$. There exist $x, y \in S$ such that $\sigma \leq x\sigma^2$ and $\sigma \leq \sigma^2 y$ as S is left and right regular, being completely regular. Then $(x, \sigma^2), (\sigma^2, y) \in X_\sigma$. Since $X_a \neq \phi$, we have

$$\begin{aligned}
\overset{-}{\lambda}(\varepsilon)(\sigma) &\leq ((\overset{-}{\lambda} \circ \overset{-}{S}) \vee (\overset{-}{S} \circ \overset{-}{\lambda}))(\varepsilon)(\sigma) \\
&= \max[(\overset{-}{\lambda} \circ \overset{-}{S})(\varepsilon)(\sigma), (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(\sigma)] \\
&= \max\left(\bigwedge_{(p_1,q_1)\in X_\sigma} \max\{\overset{-}{\lambda}(\varepsilon)(p_1), \overset{-}{S}(\varepsilon)(q_1)\}, \bigwedge_{(r_1,s_1)\in X_\sigma} \max\{\overset{-}{S}(\varepsilon)(r_1), \overset{-}{\lambda}(\varepsilon)(s_1)\}\right) \\
&\leq \max[\max\{\overset{-}{\lambda}(\varepsilon)(\sigma^2), \overset{-}{S}(\varepsilon)(y)\}, \max\{\overset{-}{S}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(\sigma^2)\}] \\
&= \max[\overset{-}{\lambda}(\varepsilon)(\sigma^2), \overset{-}{\lambda}(\varepsilon)(\sigma^2)] \\
&= \overset{-}{\lambda}(\varepsilon)(\sigma^2) \leq \max\{\overset{-}{\lambda}(\varepsilon)(\sigma), \overset{-}{\lambda}(\varepsilon)(\sigma)\} = \overset{-}{\lambda}(\varepsilon)(\sigma).
\end{aligned}$$

Similarly, we have

$$\overset{+}{\lambda}(\varepsilon)(\sigma) \quad \geq \quad \overset{+}{\lambda}(\varepsilon)(\sigma^2) \geq \min\{\overset{+}{\lambda}(\varepsilon)(\sigma), \overset{+}{\lambda}(\varepsilon)(\sigma)\} = \overset{+}{\lambda}(\varepsilon)(\sigma).$$

Thus, we obtain

$$\overset{+}{\lambda}(\varepsilon)(\sigma) = \overset{+}{\lambda}(\varepsilon)(\sigma^2), \quad \overset{-}{\lambda}(\varepsilon)(\sigma) = \overset{-}{\lambda}(\varepsilon)(\sigma^2).$$

Therefore, Assertion (i) is satisfied.

Conversely, let $\sigma \in S$. Consider the quasi-ideal $Q(\sigma^2)$ of S generated by σ^2, i.e., the set $Q(\sigma^2) = (\sigma^2 \cup (\sigma^2 S \cap S\sigma^2)]$. Then, by Theorem 2, we have $\overset{Q(\sigma^2)}{\chi_A}$ is an FBS quasi-ideal over S. By the hypothesis, we have

$$\overset{+}{\chi}_{Q(\sigma^2)}(\varepsilon)(\sigma) = \overset{+}{\chi}_{Q(\sigma^2)}(\varepsilon)(\sigma^2), \quad \overset{-}{\chi}_{Q(\sigma^2)}(\varepsilon)(\sigma) = \overset{-}{\chi}_{Q(\sigma^2)}(\varepsilon)(\sigma^2).$$

Since $\sigma^2 \in Q(\sigma^2)$, then

$$\overset{+}{\chi}_{Q(\sigma^2)}(\varepsilon)(\sigma^2) = 1, \quad \overset{-}{\chi}_{Q(\sigma^2)}(\varepsilon)(\sigma^2) = 0.$$

Thus, we get

$$\overset{+}{\chi}_{Q(\sigma^2)}(\varepsilon)(\sigma) = 1, \quad \overset{-}{\chi}_{Q(\sigma^2)}(\varepsilon)(\sigma) = 0,$$

which implies that $\sigma \in Q(\sigma^2) = (\sigma^2 \cup (\sigma^2 S \cap S\sigma^2)]$. Then, either $\sigma \leq \sigma^2$ or $\sigma \leq \sigma^2 x$ and $\sigma \leq y\sigma^2$ for some x, y in S. If $\sigma \leq \sigma^2$, then $\sigma \leq \sigma^2 = \sigma\sigma \leq \sigma^2\sigma^2 = \sigma\sigma\sigma^2 \leq \sigma^2\sigma\sigma^2 \in \sigma^2 S\sigma^2$, which implies that $\sigma \in (\sigma^2 S\sigma^2)]$. If $\sigma \leq \sigma^2 x$ and $\sigma \leq y\sigma^2$, then

$\sigma \leq (\sigma^2 x)(y\sigma^2) = \sigma^2(xy)\sigma^2 \in \sigma^2 S\sigma^2$ which implies that $\sigma \in (\sigma^2 S\sigma^2)]$. Thus, by Lemma 8.3, we have S is completely regular. □

Definition 16. *[4] An FBS set λ_A over S is called FBS semiprime iff*

$$\overset{+}{\lambda}(\epsilon)(\sigma) \geq \overset{+}{\lambda}(\epsilon)(\sigma^2), \quad \overset{-}{\lambda}(\epsilon)(\sigma) \leq \overset{-}{\lambda}(\epsilon)(\sigma^2),$$

for all $\sigma \in S$ and $\epsilon \in A$.

Definition 17. *An FBS quasi-ideal λ_A over S is called FBS semiprime quasi-ideal iff λ_A is FBS semiprime.*

In the light of Definitions 16, 17 and Proposition 5.4, we formulate the following result:

Proposition 8.4. *Let $a \leq a^2$ for all $a \in S$. Then every FBS quasi-ideal λ_A over S is an FBS semiprime quasi-ideal over S.*

Similarly, in the light of Definitions 16, 17 and Theorem 7, we establish the following theorem that characterizes completely regular ordered semigroups in terms of their FBS quasi-ideals and FBS semiprime quasi-ideals.

Theorem 8. *The following assertions are equivalent on S:*

(i) *S is completely regular.*
(ii) *Every FBS quasi-ideal over S is an FBS semiprime quasi-ideal.*

9. Conclusion

The existing literature contains several extensions of Zadeh's fuzzy set theory. All these theories have found many applications in the domain of mathematics and elsewhere. The notion of FBS sets is another important extension of fuzzy set theory. In this paper, we extend the concept of FBS sets to ordered semigroup theory and introduce the notion of FBS quasi-ideals. First some fundamental characteristics of the structure are examined and, most importantly, FBS quasi-ideals over ordered semigroups are linked with the crisp quasi-ideals. Thereafter, weakly-regular ordered semigroups are characterized by means of their FBS quasi-ideals while the ordered semigroups that are both intra-regular and left weakly-regular are characterized in terms of their FBS left (right) ideals and FBS quasi-ideals. Finally, left and right simple ordered semigroups are characterized by their FBS quasi-ideals. We also introduce FBS semiprime quasi-ideals in ordered semigroup theory and characterize completely regular ordered semigroups by their FBS (semiprime) quasi-ideals. To extend this research work, one can study the properties of FBS sets in other algebraic and non-algebraic structures such as (ordered) ring theory, (ordered) semiring theory, (ordered) hypersemigroups, lattice theory and (partial) metric spaces. In particular, ideal theory in terms of FBS sets may be studied in (ordered) ring theory, (ordered) semiring theory and (ordered) hypersemigroups.

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