Punjab University Journal of Mathematics (2022),54(10),645-657
https://doi.org/10.52280/pujm.2022.541002

Uniformly convex and starlike functions related with Liu-Owa integral operator

Syed Ghoos Ali Shah[1], Saqib Hussain[1], F. Müge Sakar[2], Akhter Rasheed[1], Muhammad Naeem[3],
[1]Department of Mathematics, Comsats University Islamabad, Abbottabad Campus, Pakistan
[2]Department of Management, Dicle University, Diyarbakır, Turkey
[3]Department of Mathematics and Statistic, International Islamic University Islamabad, Pakistan
E-mail: alishahsyedghoos@gmail.com, saqib_math@yahoo.com, mugesakar@hotmail.com, akhter_rasheed77@yahoo.com, naeem.phdma75@iiu.edu.pk,

Received: 01 June, 2021 / Accepted: 24 October, 2022 / Published online: 31 October, 2022

Abstract.: In this present paper, we introduce and explore certain new classes of uniformly convex and starlike functions related with Liu-Owa integral operator. We study explore various properties and characteristics, such as coefficient estimates, growth rate function property and partial sums. It is important to mention that our results are generalization of number of existing results in literature.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Analytic functions, integral operator, uniformly convex and starlike function.

1. Introduction

Let $\mathbb{C}$ denote the complex plane and assume that A_p denotes the class of p-valent function of the form:

$$\lambda(\omega) = \omega^p + \sum_{t=1}^{\infty} a_{t+p}\omega^{t+p}, \quad (\, p \in \mathbb{N} = \{1, 2, ...\}) \tag{1. 1}$$

which are analytic in the open unit disc $\mathbb{U} = \{\omega : \omega \in \mathbb{C} \text{ and } |\omega| < 1\}$. For $p = 1$, we denote $A_1 = A$, which is will known class of one univalent function.

By U, K and S, we mean the subclasses of A_1 which consist of all univalent, convex and starlike functions, respectively while by $S(\alpha)$ we denote the class of starlike function of order α, $\alpha \in [0, 1)$. In 1991, Goodman [[9]-[10]] introduced the classes UST and UCV of uniformly starlike and uniformly convex functions, respectively. A function $\lambda(\omega)$ is uniformly starlike (uniformly convex) in $\mathbb{U}$ if $\lambda(\omega)$ is in UST (UCV) and has the property that for every circular arc γ contained in $\mathbb{U}$, the arc $\lambda(\gamma)$ is starlike (convex).

A more one-variable representation of UST and UCV was given in [8], [11] see also [[18]-[23]].

$$\lambda \in UST \Longleftrightarrow \left| \frac{\omega\lambda^{'}(\omega)}{\lambda(\omega)} - 1 \right| \leq \mathrm{Re}\left(\frac{\omega\lambda^{'}(\omega)}{\lambda(\omega)} \right), \quad (\omega \in \mathbb{U}), \tag{1. 2}$$

and

$$\lambda \in UCV \Longleftrightarrow \left| \frac{\omega\lambda^{''}(\omega)}{\lambda^{'}(\omega)} \right| \leq \mathrm{Re}\left(1 + \frac{\omega\lambda^{''}(\omega)}{\lambda^{'}(\omega)} \right), \quad (\omega \in \mathbb{U}). \tag{1. 3}$$

In 1999, for $k \geq 0$, Kanas and Wisniowska [14] introduced the classes $k - UST$ and $k - UCV$ of uniformly convex and uniformly starlike functions, respectively.

Let $k - UST(\alpha, \beta)$ denotes the subclass of A_p consisting of functions of the form (1. 1) and satisfy the following inequality:

$$\mathrm{Re}\left(\frac{\omega\lambda^{'}(\omega)}{\lambda(\omega)} - \alpha \right) > k \left| \frac{\omega\lambda^{'}(\omega)}{\lambda(\omega)} - \beta \right|, \ (0 \leq \alpha < \beta \leq p;\ k(p-\beta) < (p-\alpha);\ \omega \in \mathbb{U}). \tag{1. 4}$$

Also, let $\lambda \in k - UCV(\alpha, \beta)$ denotes the subclass of A_p consisting of functions of the form (1. 1) and satisfy the following inequality:

$$\mathrm{Re}\left(1 + \frac{\omega\lambda^{''}(\omega)}{\lambda^{'}(\omega)} - \alpha \right) > k \left| 1 + \frac{\omega\lambda^{''}(\omega)}{\lambda^{'}(\omega)} - \beta \right|, \ \ (0 \leq \alpha < \beta \leq p;\ k(p-\beta) < (p-\alpha);\ \omega \in \mathbb{U}). \tag{1. 5}$$

It follows from (1. 4) and (1. 5) that

$$\lambda \in k - UCV(\alpha, \beta) \Longleftrightarrow \omega\lambda^{'} \in k - UST(\alpha, \beta).$$

Notice that, $0 - UST(\alpha, \beta) = S(\alpha)$ and $0 - UCV(\alpha, 0) = K(\alpha)$, where $S(\alpha)$ and $K(\alpha)$ are respectively the popular classes of starlike and convex functions of order α $(0 \leq \alpha < 1)$ in [13]. The class $k - UST(\alpha, \beta)$ denote the class of k uniformly starlike functions of order α and type β and the class $k - UCV(\alpha, \beta)$ denotes the class of k uniformly convex functions of order α and type β.

The Convolution (Hadamard product) for two functions $\lambda, \delta \in A_p$, is defined by

$$\lambda(\omega) * \delta(\omega) = \omega^p + \sum_{t=0}^{\infty} a_{t+p} b_{t+p} \omega^{t+p}.$$

where λ is given in (1. 1) and $\delta(\omega) = \omega^p + \sum\limits_{t=0}^{\infty} b_{t+p}\omega^{t+p}$.

In 2004, Liu and Owa [17] (see also [[1]-[6]], [22], [24], [26]) introduced the integral operator $G_{b,p}^{a}$: $A_p \longrightarrow A_p$ as follows:

$$G_{b,p}^{a}\lambda(\omega) = \binom{p+a+b-1}{p+b-1} \frac{a}{\omega^b} \int_0^{\omega} (1 - \frac{x}{\omega})^{a-1} x^{b-1} \lambda(x)\, dx \quad (a > 0; b > -1; p \in \mathbb{N}), \tag{1. 6}$$

and $G_{b,p}^{0}$ is the identity of A_p $\quad (a = 0; b > -1)$.

For $\lambda \in A_p$, given by (1. 1), and using properties of gamma function, we obtain

$$G_{b,p}^{a}\lambda(\omega) = \omega^{p} + \frac{\Gamma(a+b+p)}{\Gamma(b+p)} \sum_{t=1}^{\infty} \frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)} a_{t+p}\omega^{t+p} \quad (a \geq 0; b > -1; p \in \mathbb{N}). \tag{1. 7}$$

2. Main results for the class $k - U(a, b, p, \alpha, \beta, \mu, \nu)$

Inspired from above mentioned works, we introduce the following class of p-valent analytic function.

Definition 2.1. *For $0 \leq \alpha < \beta \leq 1$, $0 \leq \nu < 1$, $\mathrm{k}(1-\beta) < (1-\alpha)$ and $0 \leq \mu < 1$, a function $\lambda \in \mathrm{A}_p$ is in class $k - U(a, b, p, \alpha, \beta, \mu, \nu)$ if and only if*

$$\operatorname{Re}\left[\frac{(1-\nu)\omega\left(G_{b,p}^{a}\lambda(\omega)\right)' + \nu\left\{\begin{array}{c}\omega\left(G_{b,p}^{a}\lambda(\omega)\right)' + (1+2\mu)\,\omega^{2}\left(G_{b,p}^{a}\lambda(\omega)\right)'' \\ +\mu\omega^{3}\left(G_{b,p}^{a}\lambda(\omega)\right)'''\end{array}\right\}}{(1-\nu)\left(G_{b,p}^{a}\lambda(\omega)\right) + \nu\left\{(\omega\left(G_{b,p}^{a}\lambda(\omega)\right)' + \mu\omega^{2}\left(G_{b,p}^{a}\lambda(\omega)\right)''\right\}} - \alpha\right]$$

$$\geq k\left|\frac{\begin{array}{c}(1-\nu)\omega\left(G_{b,p}^{a}\lambda(\omega)\right)' + \nu(\omega\left(G_{b,p}^{a}\lambda(\omega)\right)' + (1+2\mu)\,\omega^{2}\left(G_{b,p}^{a}\lambda(\omega)\right)'' \\ +\mu\omega^{3}\left(G_{b,p}^{a}\lambda(\omega)\right)'''\end{array}}{(1-\nu)\left(G_{b,p}^{a}\lambda(\omega)\right) + \nu\left\{(\omega\left(G_{b,p}^{a}\lambda(\omega)\right)' + \mu\omega^{2}\left(G_{b,p}^{a}\lambda(\omega)\right)''\right\}} - \beta\right|. \tag{2. 8}$$

We also denote $k - \pounds U_\eta(a, b, p, \alpha, \beta, \mu, \nu) = k - U(a, b, p, \alpha, \beta, \mu, \nu) \cap \pounds_\eta$, where $\pounds_\eta$ the class of functions $\lambda \in \mathrm{A}_p$ of the form (1. 1) for which $arg(a_t) = \pi + (t-1)\,\eta$.

Special Cases.

Specializing parameters, $a, b, p, \alpha, \beta, \mu$ and ν, we obtain the following subclasses studied by various authors:

(1) $k - U(0, b, p, \alpha, \beta, \mu, 1) = k - U(\alpha, \beta, \mu)$, [20].
(2) $k - \pounds U_0(0, b, p, \alpha, \beta, \mu, 1) = k - \pounds U_\eta(\alpha, \beta, \mu, \nu)$, [20].
(3) $0 - \pounds U_0(0, b, p, \alpha, 1, 0, 1) = CV(\alpha, 1)$, [25].
(4) $k - \pounds U_0(0, b, p, \alpha, 1, 0, 1) = k - UCV(\alpha, 1)$, [15].
(5) $1 - U(0, b, p, \alpha, 1, 0, 1) = UCV(\alpha, 1)$, [21].
(6) $k - U(0, b, p, \alpha, \beta, 0, 0) = k - UST(\alpha, \beta)$, [7].

2.2. Coefficient Estimates.

In this section, we obtain a necessary and sufficient condition for functions $\lambda(\omega)$ in the classes $k - \pounds U_\eta(a, b, p, \alpha, \beta, \mu, \nu)$.

Theorem 2.3. *A function $\lambda(\omega)$ given by (1. 1) is in the class $k-U(a,b,p,\alpha,\beta,\mu,\nu)$ if*

$$\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\left|a_{t+p}\right|\leq C_p(1+k)-D_p(\alpha+k\beta), \tag{2. 9}$$

where

$$C_{t+p}=1-\nu+\nu(t+p)(1+\mu(t+p-1)), \tag{2. 10}$$

$$D_{t+p}=t+p+\nu(t+p)(t+p-1)(1+\mu(t+p)), \tag{2. 11}$$

$$C_p=p+\nu p(p-1)(1+\mu p), \tag{2. 12}$$

$$D_p=1-\nu+\nu p(1+\mu(p-1)), \tag{2. 13}$$

and

$$-1\leq\alpha<\beta\leq 1,\ 0\leq\mu<1,\ k(1-\beta)<1-\alpha,\ a\geq 0,\ b>-1,\ p\in\mathbb{N}\text{ and }\omega\in\mathbb{U}.$$

Proof. It suffices to show that the inequality (2. 8) hold true. As we know

$$\operatorname{Re}(\varpi)>k\left|\varpi-\beta\right|+\alpha\Longleftrightarrow\operatorname{Re}\left[(1+ke^{i\theta})\varpi-\beta ke^{i\theta}\right]>\alpha,$$

then inequality (2. 8) may be written as

$$\operatorname{Re}\left[\begin{array}{c}(1+ke^{i\theta})\left\{\frac{(1-\nu)\omega\left(G^a_{b,p}\lambda(\omega)\right)'+\nu(\omega\left(G^a_{b,p}\lambda(\omega)\right)'+(1+2\mu)\omega^2\left(G^a_{b,p}\lambda(\omega)\right)''+\mu\omega^3\left(G^a_{b,p}\lambda(\omega)\right)'''}{(1-\nu)\left(G^a_{b,p}\lambda(\omega)\right)+\nu\ \ (\omega\left(G^a_{b,p}\lambda(\omega)\right)'+\mu\omega^2\left(G^a_{b,p}\lambda(\omega)\right)''}\right\}\\ -\beta ke^{i\theta}\end{array}\right]\geq\alpha,$$

which can be written as;

$$\operatorname{Re}\left(\frac{A(\omega)}{B(\omega)}\right)\geq\alpha,$$

where

$$A(\omega)=(1+ke^{i\theta})\left\{\begin{array}{c}(1-\nu)\omega\left(G^a_{b,p}\lambda(\omega)\right)'+\nu(\omega\left(G^a_{b,p}\lambda(\omega)\right)'\\ +(1+2\mu)\,\omega^2\left(G^a_{b,p}\lambda(\omega)\right)''+\mu\omega^3\left(G^a_{b,p}\lambda(\omega)\right)'''\end{array}\right\}$$
$$-\beta ke^{i\theta}\left[(1-\nu)\left(G^a_{b,p}\lambda(\omega)\right)+\nu\left\{(\omega\left(G^a_{b,p}\lambda(\omega)\right)'+\mu\omega^2\left(G^a_{b,p}\lambda(\omega)\right)''\right\}\right],$$

and

$$B(\omega)=(1-\nu)\left(G^a_{b,p}\lambda(\omega)\right)+\nu\left\{(\omega\left(G^a_{b,p}\lambda(\omega)\right)'+\mu\omega^2\left(G^a_{b,p}\lambda(\omega)\right)''\right\}.$$

Then we have

$$\left|A(\omega)+(1-\alpha)B(\omega)\right|-\left|A(\omega)-(1+\alpha)B(\omega)\right|\geq 0.$$

Now

$$|A(\omega)+(1-\alpha)B(\omega)| = \left| \begin{array}{c} \left[C_p+(1-\alpha)D_p+ke^{i\theta}(C_p-\beta D_p)\right]\omega^p - \frac{\Gamma(a+b+p)}{\Gamma(b+p)} \\ \times\sum_{t=1}^{\infty}\left[ke^{i\theta}(\beta C_{t+p}-D_{t+p})-D_{t+p}-(1-\alpha)C_{t+p}\right] \\ \times\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}a_{t+p}\omega^{t+p} \end{array} \right|$$

$$\geq \left[ke^{i\theta}(\beta D_p-C_p)-C_p-(1-\alpha)D_p\right]\omega^p - \frac{\Gamma(a+b+p)}{\Gamma(b+p)}$$

$$\times\sum_{t=1}^{\infty}\left[ke^{i\theta}(\beta C_{t+p}-D_{t+p})-D_{t+p}-(1-\alpha)C_{t+p}\right]$$

$$\times\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\left|a_{t+p}\right|\left|\omega^{t+p}\right|, \qquad (2.\ 14)$$

and also

$$|A(\omega)-(1+\alpha)B(\omega)| = \left| \begin{array}{c} \left[C_p-(1+\alpha)D_p+ke^{i\theta}(C_p-\beta D_p)\right]\omega^p + \frac{\Gamma(a+b+p)}{\Gamma(b+p)} \\ \times\sum_{t=1}^{\infty}\left[ke^{i\theta}(\beta C_{t+p}-D_{t+p})-D_{t+p}+(1+\alpha)C_{t+p}\right] \\ \times\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}a_{t+p}\omega^{t+p} \end{array} \right|$$

$$\leq \left[ke^{i\theta}(C_p-\beta D_p)+C_p-(1+\alpha)D_p\right]\omega^p - \frac{\Gamma(a+b+p)}{\Gamma(b+p)}$$

$$\times\sum_{t=1}^{\infty}\left[ke^{i\theta}(\beta C_{t+p}-D_{t+p})-D_{t+p}+(1+\alpha)C_{t+p}\right]$$

$$\times\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\left|a_{t+p}\right|\left|\omega^{t+p}\right|. \qquad (2.\ 15)$$

Using (2. 14) and (2. 15), then we can obtain the following inequality:

$$|A(\omega)+(1-\alpha)B(\omega)| - |A(\omega)-(1+\alpha)B(\omega)|$$

$$\geq \left[ke^{i\theta}\left\{(\beta D_p-C_p)-(C_p-\beta D_p)\right\}-C_p-(1-\alpha)D_p-C_p+(1+\alpha)D_p\right]\omega^p - \frac{\Gamma(a+b+p)}{\Gamma(b+p)}$$

$$\times\sum_{t=1}^{\infty}\left[ke^{i\theta}\left\{(\beta C_{t+p}-D_{t+p})-(\beta C_{t+p}-D_{t+p})\right\}-D_{t+p}-(1-\alpha)C_{t+p}+D_{t+p}-(1+\alpha)C_{t+p}\right]$$

$$\times\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\left|a_{t+p}\right|\left|\omega^{t+p}\right|.$$

The last expression is bounded below by 0 if

$$\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\left|a_{t+p}\right| \leq C_p(1+k)-D_p(\alpha+k\beta),$$

which complete the proof. □

Theorem 2.4. *Let $\lambda(\omega)$ be given by (1. 1) and in $\pounds_\eta$; then $\lambda\in k-\pounds U_\eta(a,b,p,\alpha,\beta,\mu,\nu)$ if and only if condition (2. 9) is satised where C_{t+p}, D_{t+p}, C_p and D_p are given by (2. 10), (2. 11), (2. 12) and (2. 13) respectively.*

Proof. In view of Theorem 2.2, we need only to show that $\lambda \in k-\pounds U_\eta(a,b,p,\alpha,\beta,\mu,\nu)$ satisfies coefficient inequality (2. 9). If $\lambda \in k-\pounds U_\eta(a,b,p,\alpha,\beta,\mu,\nu)$ then by definition, we have

$$\mathrm{Re}\left[\frac{(C_p-\alpha D_p)+\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}(D_{t+p}-\alpha C_{t+p})\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}|a_{t+p}|\,\omega^{t+p-1}}{1+\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}C_{t+p}\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}|a_{t+p}|\,\omega^{t+p-1}}\right]$$
$$\geq k\left|\frac{(C_p-\beta D_p)+\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}(D_{t+p}-\beta C_{t+p})\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}|a_{t+p}|\,\omega^{t+p-1}}{1+\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}C_{t+p}\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}|a_{t+p}|\,\omega^{t+p-1}}\right|.$$

Since λ is a function of the form (1. 1) with the argument properties given in the class $\pounds_\eta$ and setting $\omega = re^{i\eta}$ in the above inequality, we have

$$\left[\frac{(C_p-\alpha D_p)-\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}(D_{t+p}-\alpha C_{t+p})\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}|a_{t+p}|\,\omega^{t+p-1}}{1-\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}C_{t+p}\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}|a_{t+p}|\,\omega^{t+p-1}}\right]$$
$$\geq k\left|\frac{(C_p-\beta D_p)+\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}(D_{t+p}-\beta C_{t+p})\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}|a_{t+p}|\,\omega^{t+p-1}}{1-\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}C_{t+p}\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}|a_{t+p}|\,\omega^{t+p-1}}\right|. \tag{2. 16}$$

Letting $r\longrightarrow 1^-$ (2. 16), leads the desired inequality

$$\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}|a_{t+p}|\leq C_p(1+k)-D_p(\alpha+k\beta).$$

The function

$$\lambda_{t.\eta}(\omega)=\omega^p-\left(\frac{\left[C_p(1+k)-D_p(\alpha+k\beta)\right]e^{i(1-t)\eta}}{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}\right)\omega^{t+p},$$
$$0\leq\eta\leq 2\pi,\quad t\geq 1, \tag{2. 17}$$

is external function. □

Corollary 2.5. *Let the function $\lambda(\omega)$ defined by (1. 1) be in the class $k-\pounds U_\eta(a,b,p,\alpha,\beta,\mu,\nu)$, then*

$$|a_{t+p}|\leq\frac{C_p(1+k)-D_p(\alpha+k\beta)}{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}\qquad(t\in N), \tag{2. 18}$$

with equality in (2. 18) is attained for the function $\lambda_{t.\eta}(\omega)$ given by (2. 17).

We now give a representation formula for elements of the class $k-\pounds U_\eta(a,b,p,\alpha,\beta,\mu,\nu)$.

Theorem 2.6. *Let $0\leq\eta\leq 2\pi$. The function $\lambda\in\mathrm{A}_p$ is in the class $k-\pounds U_\eta(a,b,p,\alpha,\beta,\mu,\nu)$ if and only if it can be represented in the form*

$$\lambda=\sum_{t=0}^{\infty}\vartheta_t\lambda_{t.\eta}, \tag{2. 19}$$

where $\lambda_{t.\eta}$ are given by (2. 17), $\vartheta_t\geq 0$ for $t\geq 0$ and $\sum_{t=0}^{\infty}\vartheta_t=1$.

Proof. Assume that

$$\lambda(\omega) = \vartheta_0\lambda_0(\omega) + \sum_{t=1}^{\infty}\vartheta_t\left[\omega^p - \left(\frac{[C_p(1+k) - D_p(\alpha+k\beta)]\,e^{i(1-t)\eta}}{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}\right)\omega^{t+p}\right]$$
$$= \sum_{t=1}^{\infty}\vartheta_t\omega^p - \sum_{t=1}^{\infty}\left(\frac{[C_p(1+k) - D_p(\alpha+k\beta)]\,e^{i(1-t)\eta}}{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}\right)\vartheta_t\omega^{t+p}.$$

Then by Theorem 2.3, $\lambda \in k - \pounds U_\eta(a,b,p,\alpha,\beta,\mu,\nu)$. It follows that

$$\lambda(\omega) = \sum_{t=1}^{\infty}\left|\frac{[C_p(1+k) - D_p(\alpha+k\beta)]\,e^{i(1-t)\eta}}{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}\right|\vartheta_t$$
$$\times\left[\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right]$$
$$= \sum_{t=1}^{\infty}[C_p(1+k) - D_p(\alpha+k\beta)]\,\vartheta_t \le (1-\vartheta_1)\,[C_p(1+k) - D_p(\alpha+k\beta)]$$
$$\le [C_p(1+k) - D_p(\alpha+k\beta)]\,.$$

Conversely assume that the function $\lambda(\omega)$ defined by (1. 1) belongs to the class $k - \pounds U_\eta(a,b,p,\alpha,\beta,\mu,\nu)$, then

$$|a_{t+p}| \le \frac{C_p(1+k) - D_p(\alpha+k\beta)}{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}, \qquad (t \in N).$$

Setting $\vartheta_t = \frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_p(1+k) - D_p(\alpha+k\beta)}\,|a_{t+p}|$, ($t \ge 1$) and $\vartheta_1 = 1 - \sum_{t=1}^{\infty}\vartheta_t$. Then $\lambda(\omega) = \sum_{t=0}^{\infty}\vartheta_t\lambda_{t.\eta}$ and this completes the proof. □

2.7. **Growth and Distortion Result.** In this section, we find a growth and distortion bound for functions $\lambda(\omega)$ in the classes $k - \pounds U_\eta(a,b,p,\alpha,\beta,\mu,\nu)$.

Theorem 2.8. *Let the function $\lambda(\omega)$ defined by (1. 1) be in the class $k - \pounds U_\eta(a,b,p,\alpha,\beta,\mu,\nu)$, then for $|\omega| = r < 1$,*

$$r^p - \left(\frac{C_p(1+k) - D_p(\alpha+k\beta)}{\left(\frac{b+p}{a+b+p}\right)[D_{p+1}(1+k) - C_{p+1}(\alpha+k\beta)]}\right)r^{p+1} \le |\lambda(\omega)|$$
$$\le r^p + \left(\frac{C_p(1+k) - D_p(\alpha+k\beta)}{\left(\frac{b+p}{a+b+p}\right)[D_{p+1}(1+k) - C_{p+1}(\alpha+k\beta)]}\right)r^{p+1}\ , \qquad (2.\ 20)$$

and

$$pr^{p-1}-\left(\frac{(p+1)\left[C_p(1+k)-D_p(\alpha+k\beta)\right]}{\left(\frac{b+p}{a+b+p}\right)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]}\right)r^p\leq\left|\lambda^{'}\left(\omega\right)\right|$$
$$\leq pr^{p-1}+\left(\frac{(p+1)\left[C_p(1+k)-D_p(\alpha+k\beta)\right]}{\left(\frac{b+p}{a+b+p}\right)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]}\right)r^p\ . \qquad (2.\ 21)$$

Proof. From Theorem (2.3), we have

$$\frac{(b+p)}{(a+b+p)}\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]\sum_{t=1}^{\infty}\left|a_{t+p}\right|$$
$$\leq\frac{\Gamma\left(a+b+p\right)}{\Gamma\left(b+p\right)}\sum_{t=1}^{\infty}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma\left(b+p+t\right)}{\Gamma\left(a+b+p+t\right)}\left|a_{t+p}\right|$$
$$\leq C_p(1+k)-D_p(\alpha+k\beta).$$

The last inequality follows from Theorem 2.3. Thus

$$\left|\lambda\left(\omega\right)\right|\leq\left|\omega\right|^p+\sum_{t=1}^{\infty}\left|a_{t+p}\right|\left|\omega\right|^{t+p}\leq r^p+r^{p+1}\sum_{t=1}^{\infty}\left|a_{t+p}\right|$$
$$\leq r^p+\left(\frac{C_p(1+k)-D_p(\alpha+k\beta)}{\left(\frac{b+p}{a+b+p}\right)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]}\right)r^{p+1}. \qquad (2.\ 22)$$

Similarly,

$$\left|\lambda\left(\omega\right)\right|\geq\left|\omega\right|^p-\sum_{t=1}^{\infty}\left|a_{t+p}\right|\left|\omega\right|^{t+p}\geq r^p-r^{p+1}\sum_{t=1}^{\infty}\left|a_{t+p}\right|$$
$$\geq r^p-\left(\frac{C_p(1+k)-D_p(\alpha+k\beta)}{\left(\frac{b+p}{a+b+p}\right)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]}\right)r^{p+1}. \qquad (2.\ 23)$$

Now, by differentiating (1. 1), we get

$$\left|\lambda^{'}\left(\omega\right)\right|\leq p\left|\omega\right|^{p-1}+\sum_{t=1}^{\infty}(t+p)\left|a_{t+p}\right|\left|\omega\right|^{t+p-1}\leq pr^{p-1}+r^p\sum_{t=1}^{\infty}(t+p)\left|a_{t+p}\right|$$
$$\leq pr^{p-1}+\left(\frac{(p+1)\left[C_p(1+k)-D_p(\alpha+k\beta)\right]}{\left(\frac{b+p}{a+b+p}\right)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]}\right)r^p, \qquad (2.\ 24)$$

and

$$\left|\lambda^{'}(\omega)\right| \geq |\omega|^{p} - \sum_{t=1}^{\infty}|a_{t+p}|\,|\omega|^{t+p} \geq pr^{p-1} - r^{p}\sum_{t=1}^{\infty}(t+p)\,|a_{t+p}| \geq$$

$$pr^{p-1} - \left(\frac{(p+1)\left[C_p(1+k) - D_p(\alpha+k\beta)\right]}{\left(\frac{b+p}{a+b+p}\right)\left[D_{p+1}(1+k) - C_{p+1}(\alpha+k\beta)\right]}\right) r^{p}. \qquad (2.\ 25)$$

Using Theorem 2.3, in (2. 25), we have

$$\frac{(b+p)}{(a+b+p)}\frac{\left[D_{p+1}(1+k) - C_{p+1}(\alpha+k\beta)\right]}{2}\sum_{t=1}^{\infty}(t+p)\,|a_{t+p}|$$
$$\leq \frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\,|a_{t+p}|$$
$$\leq C_p(1+k) - D_p(\alpha+k\beta),$$

or, equivalently

$$\sum_{t=1}^{\infty}(t+p)\,|a_{t+p}| \leq \frac{2\,(a+b+p)\left[C_p(1+k) - D_p(\alpha+k\beta)\right]}{(b+p)\left[D_{p+1}(1+k) - C_{p+1}(\alpha+k\beta)\right]}. \qquad (2.\ 26)$$

Using (2. 26) into (2. 24) and (2. 25) yields the inequality (2. 21). □

2.9. **Radii of close-to-convexity, Starlikeness and Convexity.** In this section, we obtain the radii of close-to-convexity, starlikeness and convexity for functions $\lambda(\omega)$ in the classes $k - \pounds U_{\eta}(a,b,p,\alpha,\beta,\mu,\nu)$.

Theorem 2.10. *Let* $\lambda \in k - \pounds U_{\eta}(a,b,p,\alpha,\beta,\mu,\nu)$*, then*

(i). $\lambda(\omega)$ *is starlike of order* $\varkappa$ $(0 \leq \varkappa < 1)$ *in the disc* $|\omega| < r_1$*, where*

$$r_1 = \inf\left[\left(\frac{2-p-\varkappa}{t+p-\varkappa}\right)\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{\left[C_p(1+k) - D_p(\alpha+k\beta)\right]}\right]^{\frac{1}{t}},$$

$t \geq 1.$

(ii). $\lambda(\omega)$ *is convex of order* $\varkappa$ $(0 \leq \varkappa < 1)$ *in the disc* $|\omega| < r_2$*, where*

$$r_2 = \inf\left[\left(\frac{p\,(2-p-\varkappa)}{(t+p)\,(t+p-\varkappa)}\right)\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{\left[C_p(1+k) - D_p(\alpha+k\beta)\right]}\right]^{\frac{1}{t}},$$

$t \geq 1.$

Each of these results are sharp for the extremal function $\lambda(\omega)$ *given by (2. 17).*

Proof. (i) Given $\lambda \in \pounds_{\eta}$, and λ is starlike of order $\varkappa$, we have

$$\left|\frac{\omega\lambda^{'}(\omega)}{\lambda(\omega)} - 1\right| < 1 - \varkappa. \qquad (2.\ 27)$$

For the left hand side of (2. 27), we have

$$\left|\frac{\omega\lambda^{'}(\omega)}{\lambda(\omega)}-1\right|\leq\frac{p-1+\sum_{t=1}^{\infty}(k+p-1\left|a_{t+p}\right|\left|\omega\right|^{t}}{1-\sum_{t=1}^{\infty}\left|a_{t+p}\right|\left|\omega\right|^{t}}.$$

The last expression is less than $1-\varkappa$ if

$$\sum_{t=1}^{\infty}\left(\frac{k+p-\varkappa}{2-p-\varkappa}\right)\left|a_{t+p}\right|\left|\omega\right|^{t}<1.$$

Using the fact that $\lambda\in k-\pounds U_{\eta}(a,b,p,\alpha,\beta,\mu,\nu)$ if and only if

$$\sum_{t=1}^{\infty}\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_{p}(1+k)-D_{p}(\alpha+k\beta)}\left|a_{t+p}\right|\leq 1.$$

We can say (2. 27) is true if

$$\left(\frac{k+p-\varkappa}{2-p-\varkappa}\right)\left|\omega\right|^{t}=\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_{p}(1+k)-D_{p}(\alpha+k\beta)}.$$

or equivalently

$$\left|\omega\right|^{t}=\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left(2-p-\varkappa\right)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{\left(k+p-\varkappa\right)\left[C_{p}(1+k)-D_{p}(\alpha+k\beta)\right]}.$$

which is required.

(ii) Using the fact that λ is convex if and only if $\omega\lambda^{'}(\omega)$ is starlike, we can prove (ii), on similar lines to the proof of (i). □

2.11. **Modified Hadamard Product.** Let the function $\lambda_j(\omega)$ $(j=1,2)$ be defined by

$$\lambda_{j}(\omega)=\omega^{p}+\sum_{t=1}^{\infty}a_{t+p,i}\omega^{t+p},\quad a_{t+p,i}\geq 0;\ i\in\mathbb{N}, \tag{2. 28}$$

then we define the modified Hadamard product of $\lambda_1(\omega)$ and $\lambda_2(\omega)$ by

$$\left(\lambda_{1}*\lambda_{2}\right)(\omega)=\omega^{p}-\sum_{t=0}^{\infty}a_{t+p,1}a_{t+p,2}\omega^{t+p}. \tag{2. 29}$$

Now, we prove the following.

Theorem 2.12. *Let $\lambda_j(\omega)$ $(j=1,2,...)$ given by (2. 28) be in the class $k-\pounds U_{\eta}(a,b,p,\alpha,\beta,\mu,\nu)$, then $(\lambda_1*\lambda_2)\in k-\pounds U_{\eta}(a,b,p,\Phi_1,\mu,\nu)$ for*

$$\Phi_{1}=\frac{\begin{array}{c}\left[C_{p}(1+k)-D_{p}(k\beta)\right]\left(\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right)^{2}\\-\left[D_{t+p}(1+k)-C_{t+p}(k\beta)\right]\left[C_{p}(1+k)-D_{p}(\alpha+k\beta)\right]^{2}\end{array}}{\begin{array}{c}D_{p}\left(\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right)^{2}\\-C_{t+p}\left[C_{p}(1+k)-D_{p}(\alpha+k\beta)\right]^{2}\end{array}}.$$

Proof. We need to prove the largest Φ_1, such that

$$\frac{[D_{t+p}(1+k)-C_{t+p}(\Phi_1+k\beta)]}{C_p(1+k)-D_p(\Phi_1+k\beta)}\,|a_{t+p,1}|\,|a_{t+p,2}|\leq 1.$$

From Theorem 2.3, we have

$$\sum_{t=1}^{\infty}\left[\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_p(1+k)-D_p(\alpha+k\beta)}\right]|a_{t+p,1}|\leq 1.$$

$$\sum_{t=1}^{\infty}\left[\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_p(1+k)-D_p(\alpha+k\beta)}\right]|a_{t+p,2}|\leq 1.$$

By Cauchy-Schwarz inequality, we have

$$\sum_{t=1}^{\infty}\left[\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_p(1+k)-D_p(\alpha+k\beta)}\right]\sqrt{a_{t+p,1}\,a_{t+p,2}}\leq 1. \tag{2. 30}$$

Thus it is sufficient to show that

$$\frac{[D_{t+p}(1+k)-C_{t+p}(\Phi_1+k\beta)]}{C_p(1+k)-D_p(\Phi_1+k\beta)}\,|a_{t+p,1}|\,|a_{t+p,2}|$$
$$\leq\left[\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_p(1+k)-D_p(\alpha+k\beta)}\right]\sqrt{a_{t+p,1}\,a_{t+p,2}}\,.$$

That is

$$\sqrt{a_{t+p,1}\,a_{t+p,2}}\leq\left[\frac{[C_p(1+k)-D_p(\Phi_1+k\beta)]\,\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_p(1+k)-D_p(\alpha+k\beta)\,[D_{t+p}(1+k)-C_{t+p}(\Phi_1+k\beta)]}\right]. \tag{2. 31}$$

Note that

$$\sqrt{a_{t+p,1}\,a_{t+p,2}}\leq\frac{C_p(1+k)-D_p(\alpha+k\beta)}{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}. \tag{2. 32}$$

Consequently, (2. 31) and (2. 32), we get

$$\frac{C_p(1+k)-D_p(\alpha+k\beta)}{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}$$
$$\leq\left[\frac{[C_p(1+k)-D_p(\Phi_1+k\beta)]\,\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\,[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)]\,\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{[D_{t+p}(1+k)-C_{t+p}(\Phi_1+k\beta)]\,[C_p(1+k)-D_p(\alpha+k\beta)]}\right].$$

or equivalently

$$\Phi_1 \leq \frac{\begin{array}{c}[C_p(1+k)-D_p(k\beta)]\left(\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right)^2\\ -\left[D_{t+p}(1+k)-C_{t+p}(k\beta)\right]\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^2\end{array}}{\begin{array}{c}D_p\left(\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right)^2\\ -C_{t+p}\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^2\end{array}} = \chi(t). \tag{2. 33}$$

Since $\chi_1(t)$ is increasing function for $t \geq 1$, letting $t = 1$ in (2. 33) we obtain

$$\Phi_1 \leq \chi_1(1) = \frac{\begin{array}{c}[C_p(1+k)-D_p(k\beta)]\left(\left(\frac{b+p}{a+b+p}\right)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]\right)^2\\ -\left[D_{p+1}(1+k)-C_{p+1}(k\beta)\right]\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^2\end{array}}{\begin{array}{c}D_p\left(\left(\frac{b+p}{a+b+p}\right)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]\right)^2\\ -C_{P+1}\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^2\end{array}}.$$

The proof of our theorem is now completed. □

Theorem 2.13. *Let $\lambda_j(\omega)$ $(j = 1, 2)$ given by (2. 28) be in the class $k-\pounds U_\eta(a, b, p, \alpha, \beta, \mu, \nu)$. If the sequence $\left\{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right\}$ is non-decreasing. Then function*

$$h(\omega) = \omega^p - \sum_{t=1}^{\infty}\left(a_{t+p,1}^2 + a_{t+p,2}^2\right)\omega^{t+p},$$

belongs to the class $k - \pounds U_\eta(a, b, p, \Phi_2, \mu, \nu)$ where

$$\Phi_2 \leq \frac{\begin{array}{c}\left[\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right]^2(C_p(1+k)-D_pk\beta)\\ -2\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^2\left[D_{t+p}(1+k)-C_{t+p}k\beta)\right]\end{array}}{D_p\left[\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right]^2-2C_{t+p}\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^2}.$$

Proof. We need to prove the largest Φ_2, such that
From Theorem 2.3, we have

$$\sum_{t=1}^{\infty}\left[\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_p(1+k)-D_p(\alpha+k\beta)}\right]^2 a_{t+p,1}^2$$
$$\leq \sum_{t=1}^{\infty}\left[\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_p(1+k)-D_p(\alpha+k\beta)}a_{t+p,1}\right]^2 \leq 1. \tag{2. 34}$$

and

$$\sum_{t=1}^{\infty}\left[\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_p(1+k)-D_p(\alpha+k\beta)}\right]^2 a_{t+p,2}^2$$

$$\leq \sum_{t=1}^{\infty}\left[\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_p(1+k)-D_p(\alpha+k\beta)}a_{t+p,2}\right]^2 \leq 1. \quad (2.\ 35)$$

It follows from (2. 34) and (2. 35) that

$$\sum_{t=1}^{\infty}\frac{1}{2}\left[\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_p(1+k)-D_p(\alpha+k\beta)}\right]^2\left(a_{t+p,1}^2+a_{t+p,2}^2\right)\leq 1. \quad (2.\ 36)$$

Therefore we need to find the largest Φ_2, such that

$$\frac{\left[D_{t+p}(1+k)-C_{t+p}(\Phi_2+k\beta)\right]}{C_p(1+k)-D_p(\Phi_2+k\beta)}\leq\frac{1}{2}\left[\frac{\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}}{C_p(1+k)-D_p(\alpha+k\beta)}\right]^2,$$

that is

$$\Phi_2\leq\frac{\begin{array}{c}\left[\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right]^2\left(C_p(1+k)-D_pk\beta\right)\\-2\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^2\left[D_{t+p}(1+k)-C_{t+p}k\beta\right]\end{array}}{D_p\left[\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right]^2-2C_{t+p}\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^2}. \quad (2.\ 37)$$

Since $\chi_2(t)$ is increasing function for $t\geq 1$, letting $t=1$ in (2. 33) we readily have

$$\Phi_2\leq\chi_2(1)=\frac{\begin{array}{c}\left[\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right]^2\left(C_p(1+k)-D_pk\beta\right)\\-2\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^2\left[D_{p+1}(1+k)-C_{p+1}k\beta\right]\end{array}}{\begin{array}{c}D_p\left[\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right]^2\\-2C_{p+1}\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^2\end{array}}.$$

The proof of our theorem is now completed. □

Conclusion.

In our current investigation, we have presented and studied thoroughly some new subclasses of $p-$valent functions related with uniformly convex and starlike functions, in connection with the Liu-Owa integral operator $G_{b,p}^{a}\lambda\left(\omega\right)$ given by (1. 6). We have obtained sufficient and necessary conditions in relation to these classes, including growth, distortion theorem and radius problem. Some special cases have been discussed as applications of our main results. The techniques and ideas of this paper may stimulate for further research in this area of knowledge.

REFERENCES

[1] M. K. Aouf and T. M. Seoudy, *Some properties of a certain subclass of multivalent analytic functions involving the Liu–Owa operator*, Comput. Math. Appl. **60**, (2010) 1525–1535.

[2] M. K. Aouf and T. M. Seoudy, *Some preserving subordination and super ordination of analytic functions involving the Liu–Owa integral operator*, Comput. Math. Appl, **62**, (2011) 3575–3580.

[3] M. K. Aouf and T. M. Seoudy, *Some preserving subordination and superordination of the Liu–Owa integral operator*, Complex Anal. Oper. Theory., **7**, (2013) 275–283.

[4] R. Bharati, R. Parvatham and A. Swaminathan, *On subclasses of uniformly convex functions and corresponding class of starlike functions,* Tamkang Journal of Mathematics, **28,** No.1 (1997), 17–32.

[5] N. E. Cho and S. Owa, *Sufficient conditions for meromorphic starlikeness and close to-convexity of order α,* Int. J. Math. Sci., **26,** No. 5 (2001) 317-319.

[6] M. Darus, S. Hussain, M. Raza and J. Sokol, *On a subclass of starlike functions,* Results in Mathematics.,**73**, (2018) 1-12.

[7] R. M. El-Ashwah, M. K. Aouf, A. A. Hassan and A. H. Hassan, *Certain new classes of analytic functions with varying arguments,* Journal of Complex Analysis., (2013) 20-35.

[8] B. A. Frasin, *Comprehensive family of uniformly analytic functions,* Tamkang J.Math., **36** No. 3 (2005) 243-254.

[9] A. W. Goodman, *On uniformly convex functions,* Annales Polonici Mathematici, **56,** No.1 (1991) 87–92.

[10] A. W. Goodman, *On uniformly starlike functions,* Journal of Mathematical Analysis and Applications, **155**, No. 2 (1991) 364–370.

[11] S. Hussain, A. Rasheed and M. Darus, *A subclasses of analytic functions related to k-uniformy convex and starlike functions*, journal of functions spaces., (2017).

[12] W. Janowski, *Some extermal problems for certain families of analytic functions,* Ann. Pol. Math. **28**, (1973) 297–326.

[13] S. Kanas, *Alternative chaacterization of the class k-UCV and realated classes of univalent function*, Serdica Math. J. **25**, (1999) 341-350.

[14] S. Kanas and A. Wisniowska, *Conic regions and k-uniform convexity,* Journal of Computational and Applied Mathematics, **105,** No. 1-2 (1999) 327–336.

[15] A. Y. Lashin, *On certain subclasses of meromorphic functions associated with certain integral operators*, Comput. Math. Appl. **59**, (2010) 524–531.

[16] J. L. Liu and H. M. Srivastava, *Classes of meromorphically multivalent functions associated with the generalized hyper geometric function*, Math. Comput. Model., **39**, (2004) 21–34.

[17] J. L. Liu and S. Owa, *Properties of certain integral operators*, Int. J.Math. Math. Sci., **3**, (2004) 69–75.

[18] J. E. Miller, *Convex meromrphic mapping and related functions,* Proc. Amer. Math. Soc., **25**, (1970) 220-228.

[19] A. Mannino, *Some inequalities concerning starlike and convex functions,* General Mathematics, **12**, No.1 (2004) 5–12.

[20] N. Magesh, *Certain sub classes of uniformly convex functions of order α and type β with varying arguments,* Journal of the Egyptian Mathematical Society, **21,** No.3 (2013) 184–189.

[21] F. Ronning, *Uniformly convex functions and a corresponding class of starlike functions, Proceedings of the American Mathematical Society,* **118,** No. 1 (1993) 189–196.

[22] S. G. A. Shah, S. Khan, S. Hussain, M. Darus", *q-Noor integral operator associated with starlike functions and q-conic domains,* "AIMS Mathematics, 7(**6**), 1084210859.

[23] K. Swaminathan and Raghavendar, *A close-to-convexi of basic hypergeometric functions using their Taylor coefficients*, J. Math. Appl., **35**, (2012) 111–125, .

[24] S. G. A. Shah, S. Hussain, S. Noor, M. Darus and I. Ahmad, *Multivalent Functions Related with an Integral Operator,* IJMMS 2021.

[25] H. Silverman, *Univalent functions with negative coefficients, Proceedings of the American Mathematical Society.,* **51**, (1975) 109–116.

[26] H. Tong and G. Deng, *MaJorization problem for two subclasses of analytic functions connected with the Liu-owa integral operator and exponential function,* journal of inequalities and applications., (2018).