Punjab University Journal of Mathematics (2022),54(11),679-688
https://doi.org/10.52280/pujm.2022.541102

Reproducing Kernel for Neumann Boundary Conditions

Gautam Patel
Department of Mathematics,
Veer Narmad South Gujarat University, Gujarat, India,
Email: gautamvpatel26@gmail.com

Kaushal Patel
Department of Mathematics,
Veer Narmad South Gujarat University, Gujarat, India,
Email: kbpatel@vnsgu.ac.in

Received: 25 February, 2021 / Accepted: 30 November, 2022 / Published online: 30 November, 2022

Abstract.: We investigate a kernel space which is a particular class of Hilbert space. We discuss various properties of the reproducing kernel. In particular, our aim to construct kernel in reproducing space of the specific function space (Sobolev space) with the inner product and norm. Also, we derive the reproducing kernel for Neumann boundary conditions.

AMS (MOS) Subject Classification Codes: 46Cxx; 46E35
Key Words: Inner product; Hilbert space; Reproducing kernel; Neumann boundary conditions.

1. Introduction

Reproducing kernels were discovered during the initial stage of the twentieth century by Zeremba [19] in that effort the center of interest on harmonic function with boundary value. This was the earliest reproducing kernel with the reproducibility proved correlated with function family. Actually, in the early establishment develop of the reproducing kernel hypothesis, almost all the works were execute by Bergman [11, 12, 13, 14, 15], and most of the kernels discussed in the 1930's and 1940's are Bergman kernels. Bergman raise the conversation of the kernels with one or several variables to the harmonic functions , and utilized to solve Laplace equation. It can be stated that this is the establishment of a particular trend of reproducing kernel. Next development of the reproducing kernel theory was pushed by Mercer [18]. He invented the positive definite property of reproducing kernel and known its as positive definite Hermition matrix:

$$\sum_{i,j=1}^{n} k(x_i, x_j)\zeta_i\zeta_j \geq 0.$$

In 1950, N. Aronszajn [4] outlined the past works and gave a systematic reproducing kernel theory and laid a good foundation for the research of each special case and greatly simplified the proof. In this theory unifying the Bergman and Marces concept of reproducing kernel development.

Subsequently, reproducing kernel theory was used by mathematician, scientist [1, 2, 3, 5, 6, 7, 8, 9, 10, 17] like to solve the theoretical problems of many special fields. In 1986, Cui [16] construct the reproducing kernel space and corresponding kernel in the Sobolev space.

Here, we review some aspect of reproducing kernel space and then construct the reproducing kernel for the inner product and norm of Sobolev space for $m = 2$ with Neumann boundary conditions.

2. Preliminaries

Definition 2.1. Consider $\mathcal{H} = \{f(\varrho) : f(\varrho) \in \mathbb{R}$ (Real numbers set) or $f(\varrho) \in \mathbb{C}$ (Complex numbers set) , ϱ is in abstract set$\}$ is endowed with $\langle f(\varrho), g(\varrho)\rangle_{\mathcal{H}}$, with respect to which $\mathcal{H}$ is a Hilbert space.
For an abstract set X, a function $\Re(\varrho, \varphi) : X \times X \to \mathbb{F}$ ($\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$)is called *the reproducing kernel* of Hilbert space $\mathcal{H}$ if its satisfies,

$$\langle f(\varrho), \Re(\varrho, \varphi)\rangle_{\mathcal{H}} = f(\varphi).$$

for each fixed $\varphi \in X$.

Lemma 2.2. I*n reproducing kernel space* $\mathcal{H}$*,* $\Re(\varrho, \varphi) = \overline{\Re(\varphi, \varrho)}$.

Proof. we have

$$\Re(\varrho, \varphi) = \langle \Re(\cdot, \varphi), \Re(\cdot, \varrho)\rangle_{\mathcal{H}} = \overline{\langle \Re(\cdot, \varrho), \Re(\cdot, \varphi)\rangle_{\mathcal{H}}} = \overline{\Re(\varphi, \varrho)}.$$

Hence, $\Re(\varrho, \varphi)$ is conjugate symmetric. □

Lemma 2.3. *The reproducing kernel* $\Re(\varrho, \varphi)$ *is unique in reproducing kernel space* $\mathcal{H}$*.*

Proof. Let $\Phi(\varrho, \varphi)$ be also reproducing kernel , then

$$\Phi(\varrho, \varphi) = \langle \Phi(\cdot, \varphi), \Re(\cdot, \varrho)\rangle_{\mathcal{H}} = \overline{\langle \Re(\cdot, \varrho), \Phi(\cdot, \varphi)\rangle_{\mathcal{H}}} = \overline{\Re(\varphi, \varrho)} = \Re(\varrho, \varphi).$$

Hence, reproducing kernel is unique. □

Lemma 2.4. *If* $\Re(\varrho, \varphi)$ *is the reproducing kernel in* $\mathcal{H}$ *, then for each* $\varrho \in X$ *,*$\Re(\varrho, \varrho) \geq 0$ *and* $\Re(\varrho, \varrho) = 0$ *if and only if* $\mathcal{H} = \{0\}$ *.*

Proof. We have

$$\Re(\varrho, \varrho) = \langle \Re(\cdot, \varrho), \Re(\cdot, \varrho)\rangle_{\mathcal{H}} = \|\Re(\cdot, \varrho)\|^2_{\mathcal{H}}.$$

Which gives $\Re(\varrho, \varrho) \geq 0$ and $\Re(\varrho, \varrho) = 0$ if and only if $\mathcal{H} = \{0\}$. □

Lemma 2.5. *Reproducing kernel* $\Re(\varrho, \varphi)$ *is a positive semi definite.*

Proof. For any complex number ζ_i,

$$\begin{aligned}\sum_{i,j=1}^{n} \overline{\zeta_i}\zeta_j \Re(\varrho_i, \varrho_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{\zeta_i}\zeta_j \langle \Re(\cdot, \varrho_i), \Re(\cdot, \varrho_j)\rangle_{\mathcal{H}} \\ &= \langle \sum_{j=1}^{n} \zeta_j \Re(\cdot, \varrho_j), \sum_{i=1}^{n} \zeta_i \Re(\cdot, \varrho_i)\rangle_{\mathcal{H}} \\ &= \langle \sum_{i=1}^{n} \zeta_i \Re(\cdot, \varrho_i), \sum_{i=1}^{n} \zeta_i \Re(\cdot, \varrho_i)\rangle_{\mathcal{H}} \\ &= \| \sum_{i=1}^{n} \zeta_i \Re(\cdot, \varrho_i)\|_{\mathcal{H}}^2 \geq 0.\end{aligned}$$

Hence, reproducing kernel is positive semi definite. □

Lemma 2.6. *For any fixed $\varrho \in X$, the linear functional $\Im(f(\varrho)) = f(\varrho)$ is bounded if and only if Hilbert space $\mathcal{H}$ is a reproducing kernel space.*

Proof. Since $\mathcal{H}$ is a reproducing kernel space, there exists a reproducing kernel $\Re(\varrho, \varphi)$.

$$\begin{aligned}|\Im(f(\varrho))| = |f(\varrho)| &= |\langle f(\cdot), \Re(\cdot, \varrho)\rangle_{\mathcal{H}}| \\ &\leq \|f(\cdot)\|_{\mathcal{H}} \|\Re(\cdot, \varrho)\|_{\mathcal{H}} \\ &= \|f(\cdot)\|_{\mathcal{H}} \sqrt{\langle \Re(\cdot, \varrho), \Re(\cdot, \varrho)\rangle_{\mathcal{H}}} \\ &= \|f(\cdot)\|_{\mathcal{H}} \sqrt{\Re(\varrho, \varrho)}.\end{aligned}$$

Therefore, $\Im(f(\varrho)) = f(\varrho)$ is bounded.
Now, for every $f(\varrho) \in \mathcal{H}$, because of linear functional, by F. Riesz theorem there exists a unique $\Re(\cdot, \varrho) \in \mathcal{H}$, whence $f(\varrho) = \Im(f(\varrho)) = \langle f(\cdot), \Re(\cdot, \varrho)\rangle_{\mathcal{H}}$.
Hence, the lemma is proved. □

3. Reproducing Kernel Space $\mathcal{W}_2^m[\alpha, \beta]$

In this section, the function space $\mathcal{W}_2^m[\alpha, \beta] = \{f(\varrho) : f^{(m-1)}(\varrho)$ is absolutely continuous, $f^{(m)}(\varrho) \in L^2[\alpha, \beta], \varrho \in [\alpha, \beta]\}$.
For any functions $f(\varrho), g(\varrho) \in \mathcal{W}_2^m[\alpha, \beta]$,

$$\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m} = \sum_{i=0}^{m-1} \left[\frac{d^i f(\alpha)}{d\varrho^i}\frac{d^i g(\alpha)}{d\varrho^i} + \frac{d^i f(\beta)}{d\varrho^i}\frac{d^i g(\beta)}{d\varrho^i}\right] + \int_\alpha^\beta \frac{d^m f(\varrho)}{d\varrho^m}\frac{d^m g(\varrho)}{d\varrho^m} d\varrho,$$

$$\|f(\varrho)\|_{\mathcal{W}_2^m} = \sqrt{\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m}}.$$

Theorem 3.1. *The space $\mathcal{W}_2^m[\alpha, \beta]$ is an inner product space.*

Proof. Let $f(\varrho), g(\varrho), h(\varrho) \in \mathcal{W}_2^m[\alpha, \beta]$.
Here,

$$\langle f(\varrho), f(\varrho)\rangle_{\mathcal{W}_2^m} = \sum_{i=0}^{m-1} \left[\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2 + \left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2\right] + \int_\alpha^\beta \left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 d\varrho.$$

Since, $\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2 > 0$ and $\left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2 > 0$,$0 \le i \le m-1$.
Also, $\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2 = 0$ and $\left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2 = 0$ if and only if $\frac{d^i f(\alpha)}{d\varrho^i} = 0$ and $\frac{d^i f(\beta)}{d\varrho^i} = 0$, $0 \le i \le m-1$.
And , $\left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 > 0$ and $\left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 = 0$ if and only if $\frac{d^m f(\varrho)}{d\varrho^m} = 0$,$\forall \varrho \in [\alpha, \beta]$.
Therefore, $\int_\alpha^\beta \left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 d\varrho > 0$ and $\int_\alpha^\beta \left(\frac{d^m f(\varrho)}{d\varrho^m}\right)^2 d\varrho = 0$ if and only if $\frac{d^m f(\varrho)}{d\varrho^m} = 0, \forall \varrho \in [\alpha, \beta]$.
Thus, $\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m}$ is positive definite.
Clearly, $\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m}$=$\langle g(\varrho), f(\varrho)\rangle_{\mathcal{W}_2^m}$ which gives $\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m}$ is symmetric.
Now for linearity consider scalars a and b,

$$\begin{aligned}
&\langle af(\varrho) + bg(\varrho), h(\varrho)\rangle_{\mathcal{W}_2^m} = \\
&\sum_{i=0}^{m-1}\left[\left(a\frac{d^i f(\alpha)}{d\varrho^i} + b\frac{d^i g(\alpha)}{d\varrho^i}\right)\frac{d^i h(\alpha)}{d\varrho^i} + \left(a\frac{d^i f(\beta)}{d\varrho^i} + b\frac{d^i g(\beta)}{d\varrho^i}\right)\frac{d^i h(\beta)}{d\varrho^i}\right] \\
&+ \int_\alpha^\beta \left[a\frac{d^m f(\varrho)}{d\varrho^m} + b\frac{d^m g(\varrho)}{d\varrho^m}\right]\frac{d^m h(\varrho)}{d\varrho^m} d\varrho \\
&= \sum_{i=0}^{m-1}\left[a\frac{d^i f(\alpha)}{d\varrho^i}\frac{d^i h(\alpha)}{d\varrho^i} + a\frac{d^i f(\beta)}{d\varrho^i}\frac{d^i h(\beta)}{d\varrho^i}\right] + \int_\alpha^\beta a\frac{d^m f(\varrho)}{d\varrho^m}\frac{d^m h(\varrho)}{d\varrho^m} d\varrho \\
&+ \sum_{i=0}^{m-1}\left[b\frac{d^i f(\alpha)}{d\varrho^i}\frac{d^i h(\alpha)}{d\varrho^i} + b\frac{d^i f(\beta)}{d\varrho^i}\frac{d^i h(\beta)}{d\varrho^i}\right] + \int_\alpha^\beta b\frac{d^m f(\varrho)}{d\varrho^m}\frac{d^m h(\varrho)}{d\varrho^m} d\varrho \\
&= a\langle f(\varrho), h(\varrho)\rangle_{\mathcal{W}_2^m} + b\langle g(\varrho), h(\varrho)\rangle_{\mathcal{W}_2^m}.
\end{aligned}$$

Thus, $\langle f(\varrho), g(\varrho)\rangle_{\mathcal{W}_2^m}$ is linear.
This completes the proof. □

Theorem 3.2. *The space $\mathcal{W}_2^m[\alpha, \beta]$ is a Hilbert space.*

Proof. Consider $f_n(\varrho)$, $n = 1, 2, \ldots$ is a Cauchy sequence in $\mathcal{W}_2^m[\alpha, \beta]$.
Therefore,

$$\begin{aligned}
\|f_{n+p} - f_n\|^2_{\mathcal{W}_2^m} = &\sum_{i=0}^{m-1}\left[\left(\frac{d^i f_{n+p}(\alpha)}{d\varrho^i} - \frac{d^i f_n(\alpha)}{d\varrho^i}\right)^2 + \left(\frac{d^i f_{n+p}(\beta)}{d\varrho^i} - \frac{d^i f_n(\beta)}{d\varrho^i}\right)^2\right] \\
&+ \int_\alpha^\beta \left(\frac{d^m f_{n+p}(\varrho)}{d\varrho^m} - \frac{d^m f_n(\varrho)}{d\varrho^m}\right)^2 d\varrho \to 0
\end{aligned}$$

as $n \to \infty$.
Which gives, $\frac{d^i f_{n+p}(\alpha)}{d\varrho^i} - \frac{d^i f_n(\alpha)}{d\varrho^i} \to 0$ as $n \to \infty$, $0 \le i \le m-1$,$n = 1, 2, \ldots$.
Similarly, $\frac{d^i f_{n+p}(\beta)}{d\varrho^i} - \frac{d^i f_n(\beta)}{d\varrho^i} \to 0$ as $n \to \infty$, $0 \le i \le m-1$,$n = 1, 2, \ldots$.
And $\int_\alpha^\beta \left(\frac{d^m f_{n+p}(\varrho)}{d\varrho^m} - \frac{d^m f_n(\varrho)}{d\varrho^m}\right)^2 d\varrho \to 0$ as $n \to \infty$.
Which indicates that for any i $(0 \le i \le m-1)$, the sequences $\frac{d^i f_n(\alpha)}{d\varrho^i}$ and $\frac{d^i f_n(\beta)}{d\varrho^i}$,$n =$

$1, 2, \ldots$ are Cauchy sequences in R and $\frac{d^m f_n(\varrho)}{d\varrho^m}$,$n = 1, 2, \ldots$ is a Cauchy sequence in space $L^2[\alpha, \beta]$.
So, there exists unique real numbers c_i and d_i, $0 \leq i \leq m-1$ and unique function $h(\varrho) \in L^2[\alpha, \beta]$ such that, $\frac{d^i f_n(\alpha)}{d\varrho^i} \to c_i$ and $\frac{d^i f_n(\beta)}{d\varrho^i} \to d_i$,$0 \leq i \leq m-1$ and $\int_\alpha^\beta \left(\frac{d^m f_n(\varrho)}{d\varrho^m} - h(\varrho)\right)^2 d\varrho \to 0$ as $n \to \infty$.
We must have $g(\varrho) \in \mathcal{W}_2^m[\alpha, \beta]$ with $\frac{d^i g(\alpha)}{d\varrho^i} = c_i$, $\frac{d^i g(\beta)}{d\varrho^i} = d_i$, $0 \leq i \leq m-1$ and $\frac{d^m g(\varrho)}{d\varrho^m} = h(\varrho)$.
Moreover,

$$\begin{aligned}
\|f_n(\varrho) - g(\varrho)\|^2_{\mathcal{W}_2^m} &= \sum_{i=0}^{m-1} \left[\left(\frac{d^i f_n(\alpha)}{d\varrho^i} - \frac{d^i g(\alpha)}{d\varrho^i} \right)^2 + \left(\frac{d^i f_n(\beta)}{d\varrho^i} - \frac{d^i g(\beta)}{d\varrho^i} \right)^2 \right] \\
&\quad + \int_\alpha^\beta \left(\frac{d^m f_n(\varrho)}{d\varrho^m} - \frac{d^m g(\varrho)}{d\varrho^m} \right)^2 d\varrho \\
&= \sum_{i=0}^{m-1} \left[\left(\frac{d^i f_n(\alpha)}{d\varrho^i} - c_i \right)^2 + \left(\frac{d^i f_n(\beta)}{d\varrho^i} - d_i \right)^2 \right] \\
&\quad + \int_\alpha^\beta \left(\frac{d^m f_n(\varrho)}{d\varrho^m} - h(\varrho) \right)^2 d\varrho \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence, the function space $\mathcal{W}_2^m$ is a Hilbert space. □

Theorem 3.3. *The space $\mathcal{W}_2^m[\alpha, \beta]$ is a reproducing kernel Hilbert space.*

Proof. As per Lemma 2.6, suppose that $\Im(f) = f(\varrho)$, $\varrho \in [\alpha, \beta]$ is linear functional of $\mathcal{W}_2^m[\alpha, \beta]$ and $f(\varrho) \in \mathcal{W}_2^m$.
We have,

$$\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}} = \frac{d^{m-1} f(\alpha)}{d\varrho^{m-1}} + \int_\alpha^\varrho \frac{d^m f(\varrho)}{d\varrho^m} d\varrho,$$

and

$$\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}} = \int_\varrho^\beta \frac{d^m f(\varrho)}{d\varrho^m} d\varrho - \frac{d^{m-1} f(\beta)}{d\varrho^{m-1}}.$$

Therefore,

$$\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}} = \frac{1}{2} \left[\frac{d^{m-1} f(\alpha)}{d\varrho^{m-1}} - \frac{d^{m-1} f(\beta)}{d\varrho^{m-1}} \right] + \frac{1}{2} \int_\alpha^\beta \frac{d^m f(\varrho)}{d\varrho^m} d\varrho.$$

Obviously,

$$\left| \frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}} \right| \leq \left| \frac{d^{m-1} f(\alpha)}{d\varrho^{m-1}} \right| + \left| \frac{d^{m-1} f(\beta)}{d\varrho^{m-1}} \right| + \int_\alpha^\beta \left| \frac{d^m f(\varrho)}{d\varrho^m} \right| d\varrho. \tag{3. 1}$$

Since,

$$\int_{\alpha}^{\beta}\left|\frac{d^m f(\varrho)}{d\varrho^m}\right|d\varrho \leq \left[(\beta-\alpha)\int_{\alpha}^{\beta}\left|\frac{d^m f(\varrho)}{d\varrho^m}\right|^2 d\varrho\right]^{\frac{1}{2}}$$
$$= K_0\left[\int_{\alpha}^{\beta}\left|\frac{d^m f(\varrho)}{d\varrho^m}\right|^2 d\varrho\right]^{\frac{1}{2}} \tag{3. 2}$$
$$\leq K_0\left[\sum_{i=0}^{m-1}\left(\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2+\left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2\right)+\int_{\alpha}^{\beta}\left|\frac{d^m f(\varrho)}{d\varrho^m}\right|^2\right]^{\frac{1}{2}}$$
$$= K_0\|f\|_{\mathcal{W}_2^m}.$$

Now, for any i, $0\leq i\leq m-1$,

$$\left|\frac{d^i f(\alpha)}{d\varrho^i}\right| \leq \left[\sum_{i=0}^{m-1}\left(\left(\frac{d^i f(\alpha)}{d\varrho^i}\right)^2+\left(\frac{d^i f(\beta)}{d\varrho^i}\right)^2\right)+\int_{\alpha}^{\beta}\left|\frac{d^m f(\varrho)}{d\varrho^m}\right|^2\right]^{\frac{1}{2}} \tag{3. 3}$$
$$= \|f\|_{\mathcal{W}_2^m}.$$

Similarly,

$$\left|\frac{d^i f(\beta)}{d\varrho^i}\right| \leq \|f\|_{\mathcal{W}_2^m}. \tag{3. 4}$$

From (3. 1) to (3. 4),

$$\left|\frac{d^{m-1} f(\varrho)}{d\varrho^{m-1}}\right| \leq K_1\|f\|_{\mathcal{W}_2^m}. \tag{3. 5}$$

Analogously,

$$\left|\frac{d^{m-2} f(\varrho)}{d\varrho^{m-2}}\right| \leq K_2\|f\|_{\mathcal{W}_2^m}.$$

Thus,$|\Im(f)| = |f(\varrho)| \leq K_m\|f\|_{\mathcal{W}_2^m}$.
Hence, $\Im$ is bounded functional which provide that $\mathcal{W}_2^m[a,b]$ is reproducing kernel Hilbert space. □

4. METHOD TO CONSTRUCT REPRODUCING KERNEL

Suppose $\Re(\varrho,\varphi)$ is the reproducing kernel function of $\mathcal{W}_2^m[\alpha,\beta]$, then for any fixed $\varphi\in[\alpha,\beta]$ and any $f(\varrho)\in\mathcal{W}_2^m[\alpha,\beta]$, $\Re(\varrho,\varphi)$ must satisfy

$$\langle f(\varrho),\Re(\varrho,\varphi)\rangle_{\mathcal{W}_2^m} = f(\varphi).$$

Therefore,

$$\langle f(\varrho),\Re(\varrho,\varphi)\rangle_{\mathcal{W}_2^m} = \sum_{i=0}^{m-1}\left[\frac{d^i f(\alpha)}{d\varrho^i}\frac{\partial^i\Re(\alpha,\varphi)}{\partial\varrho^i}+\frac{d^i f(\beta)}{d\varrho^i}\frac{\partial^i\Re(\beta,\varphi)}{\partial\varrho^i}\right] + \int_{\alpha}^{\beta}\frac{d^m f(\varrho)}{d\varrho^m}\frac{\partial^m\Re(\varrho,\varphi)}{\partial\varrho^m}d\varrho. \tag{4. 6}$$

Since,

$$\int_{\alpha}^{\beta} \frac{d^m f(\varrho)}{d\varrho^m} \frac{\partial^m \Re(\varrho,\varphi)}{\partial \varrho^m} d\varrho = \sum_{i=0}^{m-1} \left((-1)^i \frac{d^{m-i-1} f(\varrho)}{d\varrho^{m-i-1}} \frac{\partial^{m+i} \Re(\varrho,\varphi)}{\partial \varrho^{m+i}} \right)_{\varrho=\alpha}^{\beta} + (-1)^m \int_{\alpha}^{\beta} f(\varrho) \frac{\partial^{2m} \Re(\varrho,\varphi)}{\partial \varrho^{2m}} d\varrho. \quad (4.\ 7)$$

Also,

$$\sum_{i=0}^{m-1} ((-1)^i \frac{d^{m-i-1} f(\varrho)}{d\varrho^{m-i-1}} \frac{\partial^{m+i} \Re(\varrho,\varphi)}{\partial \varrho^{m+i}} = \sum_{i=0}^{m-1} (-1)^{m-i-1} \frac{d^i f(\varrho)}{d\varrho^i} \frac{\partial^{2m-i-1} \Re(\varrho,\varphi)}{\partial \varrho^{2m-i-1}}. \quad (4.\ 8)$$

From equations (4. 6) to (4. 8), we get

$$\begin{aligned} \langle f(\varrho), \Re(\varrho,\varphi) \rangle_{\mathcal{W}_2^m} = \sum_{i=0}^{m-1} \Bigg[\frac{d^i f(\alpha)}{d\varrho^i} \left(\frac{\partial^i \Re(\alpha,\varphi)}{\partial \varrho^i} - (-1)^{m-i-1} \frac{\partial^{2m-i-1} \Re(\alpha,\varphi)}{\partial \varrho^{2m-i-1}} \right) \\ + \frac{d^i f(\beta)}{d\varrho^i} \left((-1)^{m-i-1} \frac{\partial^{2m-i-1} \Re(\beta,\varphi)}{\partial \varrho^{2m-i-1}} + \frac{\partial^i \Re(\beta,\varphi)}{\partial \varrho^i} \right) \Bigg] \\ + (-1)^m \int_{\alpha}^{\beta} f(\varrho) \frac{\partial^{2m} \Re(\varrho,\varphi)}{\partial \varrho^{2m}} d\varrho. \end{aligned} \quad (4.\ 9)$$

Now, from equations (4. 6), (4. 9) and the Dirac delta function

$$(-1)^m \frac{\partial^{2m} \Re(\varrho,\varphi)}{\partial \varrho^{2m}} = \delta(\varrho - \varphi), \quad (4.\ 10)$$

$$\frac{\partial^i \Re(\alpha,\varphi)}{\partial \varrho^i} - (-1)^{m-i-1} \frac{\partial^{2m-i-1} \Re(\alpha,\varphi)}{\partial \varrho^{2m-i-1}} = 0, 0 \le i \le m-1, \quad (4.\ 11)$$

$$(-1)^{m-i-1} \frac{\partial^{2m-i-1} \Re(\beta,\varphi)}{\partial \varrho^{2m-i-1}} + \frac{\partial^i \Re(\beta,\varphi)}{\partial \varrho^i} = 0, 0 \le i \le m-1. \quad (4.\ 12)$$

Here, $\Re(\varrho,\varphi)$ is the solution of the following constant coefficient $2m$ order differential equation with boundary conditions (4. 11) and (4. 12)

$$(-1)^m \frac{\partial^{2m} \Re(\varrho,\varphi)}{\partial \varrho^{2m}} = 0. \quad (4.\ 13)$$

The equation (4. 13) has characteristic equation $\lambda^{2m} = 0$ whose the characteristic root $\lambda = 0$ with multiplicity $2m$.
Therefore,

$$\Re(\varrho,\varphi) = \begin{cases} \Re_1(\varrho,\varphi) = \sum_{i=1}^{2m} c_i(\varphi) \varrho^{i-1}, & \varrho < \varphi, \\ \Re_2(\varrho,\varphi) = \sum_{i=1}^{2m} d_i(\varphi) \varrho^{i-1}, & \varrho > \varphi. \end{cases} \quad (4.\ 14)$$

Since, the solution (4. 14) of (4. 13) also satisfied the following conditions

$$\frac{\partial^i \Re_1(\varphi,\varphi)}{\partial \varrho^i} = \frac{\partial^i \Re_2(\varphi,\varphi)}{\partial \varrho^i}, 0 \le i \le 2m-2, \quad (4.\ 15)$$

$$\frac{\partial^{2m-1} \Re_1(\varphi^+,\varphi)}{\partial \varrho^{2m-1}} - \frac{\partial^{2m-1} \Re_2(\varphi^-,\varphi)}{\partial \varrho^{2m-1}} = \frac{1}{(-1)^m}. \quad (4.\ 16)$$

Using boundary conditions (4. 11), (4. 12), (4. 15) and (4. 16), we can derive the reproducing kernel $\Re(\varrho, \varphi)$ for any m.

5. Reproducing Kernel For Neumann Boundary Conditions

In mathematics, there are many mathematical formulation from real world problem either in ordinary differential equations or in partial differential equations with Neumann boundary conditions. To solve these type of problems using reproducing kernel, we have need reproducing kernel which satisfied Neumann boundary conditions. so that, in this section, we derived reproducing kernel for $m = 2$ with Neumann boundary conditions.
Therefore, the function space $\mathcal{W}_2^2[\alpha, \beta]$ is defined as, $\mathcal{W}_2^2[\alpha, \beta] = \{f(\varrho) : f(\varrho), f'(\varrho)$ are absolutely continuous,$f''(\varrho) \in L^2[\alpha, \beta], \varrho \in [\alpha, \beta], f'(\alpha) = 0, f'(\beta) = 0\}$.
Here, we considered $m = 2$ in (4. 13) of Section 4 with the two Neumann boundary conditions so we obtained fourth order differential equation

$$\frac{\partial^4 \Re(\varrho, \varphi)}{\partial \varrho^4} = 0. \tag{5. 17}$$

We know that equation (5. 17) has characteristic equation $\lambda^4 = 0$, and the characteristic value $\lambda = 0$ is a root whose multiplicity is four. Therefore the general solution of equation (5. 17) is

$$\Re(\varrho, \varphi) = \begin{cases} \Re_1(\varrho, \varphi) = c_1(\varphi) + c_2(\varphi)\varrho + c_3(\varphi)\varrho^2 + c_4(\varphi)\varrho^3, & \varrho \leq \varphi, \\ \Re_2(\varrho, \varphi) = d_1(\varphi) + d_2(\varphi)\varrho + d_3(\varphi)\varrho^2 + d_4(\varphi)\varrho^3, & \varrho > \varphi. \end{cases} \tag{5. 18}$$

Now we are ready to obtain the coefficients $c_1(\varphi), c_2(\varphi), c_3(\varphi), c_4(\varphi), d_1(\varphi), d_2(\varphi), d_3(\varphi)$ and $d_4(\varphi)$.
From the equations (4. 15) , (4. 16) and two Neumann boundary conditions, we get the boundary conditions for the differential equation (5. 17),

$$\begin{aligned}
&\Re(\alpha, \varphi) + \frac{\partial^3 \Re(\alpha, \varphi)}{\partial \varrho^3} = 0, \\
&\frac{\partial \Re(\alpha, \varphi)}{\partial \varrho} = 0, \\
&\Re(\beta, \varphi) - \frac{\partial^3 \Re(\beta, \varphi)}{\partial \varrho^3} = 0, \\
&\frac{\partial \Re(\beta, \varphi)}{\partial \varrho} = 0, \\
&\Re_1(\varphi, \varphi) = \Re_2(\varphi, \varphi), \\
&\frac{\partial \Re_1(\varphi, \varphi)}{\partial \varrho} = \frac{\partial \Re_2(\varphi, \varphi)}{\partial \varrho}, \\
&\frac{\partial^2 \Re_1(\varphi, \varphi)}{\partial \varrho^2} = \frac{\partial^2 \Re_2(\varphi, \varphi)}{\partial \varrho^2}, \\
&\frac{\partial^3 \Re_1(\varphi^+, \varphi)}{\partial \varrho^3} - \frac{\partial^3 \Re_2(\varphi^-, \varphi)}{\partial \varrho^3} = 1.
\end{aligned} \tag{5. 19}$$

From equations (5. 18) and (5. 19), we get the eight linear equations with variables $c_1(\varphi), c_2(\varphi), c_3(\varphi), c_4(\varphi), d_1(\varphi), d_2(\varphi), d_3(\varphi)$ and $d_4(\varphi)$. Using any linear algebra method, we get the coefficients

$$\begin{aligned} c_1(\varphi) =& \frac{\alpha^2\,\varphi}{2} - \frac{\alpha^3}{3} - \frac{\left(-\alpha^3+3\,\beta\,\alpha^2+12\right)\,\left(-\alpha^3+3\,\beta\,\alpha^2+3\,\alpha\,\varphi^2-6\,\beta\,\alpha\,\varphi-2\,\varphi^3+3\,\beta\,\varphi^2+12\right)}{12\,\left(-\alpha^3+3\,\alpha^2\,\beta-3\,\alpha\,\beta^2+\beta^3+24\right)} \\ &+ \frac{\alpha^2\,(\alpha-\varphi)^2}{4\,(\alpha-\beta)}+1, \end{aligned}$$

$$c_2(\varphi) = -\frac{(\beta-\varphi)^2}{2} - \frac{\beta\,(\beta-\varphi)^2}{2\,(\alpha-\beta)} - \frac{\alpha\,\beta\,\left(\beta^3-3\,\alpha\,\beta^2-3\,\beta\,\varphi^2+6\,\alpha\,\beta\,\varphi+2\,\varphi^3-3\,\alpha\,\varphi^2+12\right)}{2\,\left(-\alpha^3+3\,\alpha^2\,\beta-3\,\alpha\,\beta^2+\beta^3+24\right)},$$

$$c_3(\varphi) = \frac{(\beta-\varphi)^2}{4\,(\alpha-\beta)} + \frac{(\alpha+\beta)\,\left(\beta^3-3\,\alpha\,\beta^2-3\,\beta\,\varphi^2+6\,\alpha\,\beta\,\varphi+2\,\varphi^3-3\,\alpha\,\varphi^2+12\right)}{4\,\left(-\alpha^3+3\,\alpha^2\,\beta-3\,\alpha\,\beta^2+\beta^3+24\right)},$$

$$c_4(\varphi) = -\frac{\beta^3-\alpha\,\left(3\,\beta^2-6\,\beta\,\varphi+3\,\varphi^2\right)-3\,\beta\,\varphi^2+2\,\varphi^3+12}{6\,\left(-\alpha^3+3\,\alpha^2\,\beta-3\,\alpha\,\beta^2+\beta^3+24\right)},$$

$$\begin{aligned} d_1(\varphi) =& \frac{\beta^3}{3} - \frac{\beta^2\,\varphi}{2} - \frac{\left(\beta^3-3\,\alpha\,\beta^2+12\right)\,\left(\beta^3-3\,\alpha\,\beta^2-3\,\beta\,\varphi^2+6\,\alpha\,\beta\,\varphi+2\,\varphi^3-3\,\alpha\,\varphi^2+12\right)}{12\,\left(-\alpha^3+3\,\alpha^2\,\beta-3\,\alpha\,\beta^2+\beta^3+24\right)} \\ &+ \frac{\beta^2\,(\beta-\varphi)^2}{4\,(\alpha-\beta)}+1, \end{aligned}$$

$$d_2(\varphi) = \frac{(\alpha-\varphi)^2}{2} - \frac{\alpha\,(\alpha-\varphi)^2}{2\,(\alpha-\beta)} + \frac{\alpha\,\beta\,\left(-\alpha^3+3\,\beta\,\alpha^2+3\,\alpha\,\varphi^2-6\,\beta\,\alpha\,\varphi-2\,\varphi^3+3\,\beta\,\varphi^2+12\right)}{2\,\left(-\alpha^3+3\,\alpha^2\,\beta-3\,\alpha\,\beta^2+\beta^3+24\right)},$$

$$d_3(\varphi) = \frac{(\beta-\varphi)^2}{4\,(\alpha-\beta)} - \frac{\varphi}{2} + \frac{(\alpha+\beta)\,\left(\beta^3-3\,\alpha\,\beta^2-3\,\beta\,\varphi^2+6\,\alpha\,\beta\,\varphi+2\,\varphi^3-3\,\alpha\,\varphi^2+12\right)}{4\,\left(-\alpha^3+3\,\alpha^2\,\beta-3\,\alpha\,\beta^2+\beta^3+24\right)},$$

$$d_4(\varphi) = \frac{3\,\alpha\,\varphi^2+\beta\,\left(3\,\alpha^2-6\,\alpha\,\varphi+3\,\varphi^2\right)-\alpha^3-2\,\varphi^3+12}{6\,\left(-\alpha^3+3\,\alpha^2\,\beta-3\,\alpha\,\beta^2+\beta^3+24\right)}.$$

6. CONCLUSION

In this paper we derived a generalized reproducing kernel for Neumann boundary conditions using an inner product. This reproducing kernel is used to solve the ordinary differential equations of the first with Neumann boundary condition explicitly for $m = 2$. The derive reproducing kernel is generalized for n^{th} order ordinary differential equations by substitute $m = n + 1$.

7. ACKNOWLEDGMENTS

I wish to thank the editor-in-chief and the anonymous referees for their valuable suggestions to improve the quality of the paper.

REFERENCES

[1] Z. Al-Zhour, *Notes and counterexamples on positive (semi) definite properties of some matrix products*, Ain Shams Engineering Journal **9**, No. 4 (2018) 3217-3220.

[2] Z. Al-Zhour, *New techniques for solving some matrix and matrix differential equations*, Ain Shams Engineering Journal **6**, No. 1 (2015) 347-354.

[3] Z. Al-Zhour, *Efficient solutions of coupled matrix and matrix differential equations*, Intelligent Control and Automation **3**, No. 2 (2012) 176-187.

[4] N. Aronszajn, *Theory of reproducing kernels*, Transactions of the American mathematical society **68**, No. 3 (1950) 337-404.

[5] O. A. Arqub, *Application of residual power series method for the solution of time-fractional Schrödinger equations in one-dimensional space*, Fundamenta Informaticae **166**, No. 2 (2019) 87-110.

[6] O. A. Arqub, *Numerical solutions of systems of first-order, two-point BVPs based on the reproducing kernel algorithm*, Calcolo **55**, No. 3 (2018) 31.

[7] O. A. Arqub, *Numerical algorithm for the solutions of fractional order systems of Dirichlet function types with comparative analysis*, Fundamenta Informaticae **166**, No. 2 (2019) 111-137.

[8] O. A. Arqub, *Adaptation of reproducing kernel algorithm for solving fuzzy Fredholm–Volterra integrodifferential equations*, Neural Computing and Applications **28**, No. 7 (2017) 1591-1610.

[9] O. A. Arqub, M. Al-Smadi, S. Momani and T. Hayat, *Application of reproducing kernel algorithm for solving second-order, two-point fuzzy boundary value problems*, Soft Computing **21**, No. 23 (2017) 7191-7206.

[10] O. A. Arqub and N. Shawagfeh, *Application of reproducing kernel algorithm for solving Dirichlet time-fractional diffusion-Gordon types equations in porous media*, Journal of Porous Media **22**, No. 4 (2019).

[11] S. Bergman, *Uber ein Verfahren zur Konstruktion der Näherungslösungen der Gleichung* $\Delta u + \tau^2 u = 0$, Anhang zur Arbeit: Uber die Knickung von rechteckigen Platten bei Schubbeanspruchung, (1936) 97-106.

[12] S. Bergman, *Über die Entwicklung der harmonischen Funktionen der Ebene und des Raumes nach Orthogonalfunktionen*, Mathematische Annalen **86**, No. 3 (1922) 238-271.

[13] S. Bergman, *Über Kurvenintegrale von Funktionen zweier komplexen Ver-änderlichen, die die Differentialgleichung* $\Delta V + V = 0$ *befriedigen*, Mathematische Zeitschrift **32**, (1930) 386-406.

[14] S. Bergman, *Sur un lien entre la théorie des équations aux derivées partielles elliptiques et celle des functions d'une variable complexe*, CR Acad. Sci. Paris **205**, (1937) 1360-1362.

[15] S. Bergman and others, *The approximation of functions satisfying a linear partial differential equation*, Duke Mathematical Journal **6**, No. 3 (1940) 537-561.

[16] M. Cui and Z. Deng, *On the best operator of interpolation*, Math. Numer. Sin **8**, (1986) 209-216.

[17] A. KiliÃ, Z. Al-Zhour and others, *On Convergents Infinite Products and Some Generalized Inverses of Matrix Sequences*, Abstract and Applied Analysis **2011**, (2011) 1-20.

[18] J. Mercer, *Functions of positive and negative type and their connection with the theory of integral equations*, Philosophical Transsaction of the Royal Society of London Ser **209**, (1909) 415-446.

[19] S. Zaremba, *Sur Le Calcul Numeric Des Fonctions Demandees Dans le Problem de Dirichlet et le Probleme hydrodynamique*, Bulletin International de LAcademie des Sciences de Cracovie, (1908) 125-195.