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OSTROWSKI-TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR $r-$TIMES DIFFERENTIABLE $h-$CONVEX FUNCTIONS

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Abstract.: The authors have tried to prove some Ostrowski-type fractional integral inequalities for $r-$times differentiable functions and generalized some results which were carried out in both [6, 25]. Some applications to special means are given. The methods and techniques of this paper could further stimulate the research conducted in the field of fractional integral inequalities.

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1. Introduction

In 1938, A. M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the derivative of a differentiable function by its integral mean as follows:

Theorem 1.1. [6] *Let $\breve{f} : I \to \mathbb{R}$, I is an interval in $\mathbb{R}$, be a differentiable function in I^o, the interior of I and $\mu, \nu \in I^o$, $\mu < \nu$. If $|\breve{f}'(u)| \leq M$ for all $u \in [\mu, \nu]$, then*

$$\left|\breve{f}(\phi) - \frac{1}{\nu - \mu}\int_\mu^\nu \breve{f}(u)du\right| \leq \left[\frac{1}{4} + \frac{\left(\phi - \frac{\mu+\nu}{2}\right)^2}{(\nu - \mu)^2}\right](\nu - \mu)M, \tag{1. 1}$$

for $\phi \in [\mu, \nu]$, considering M as a constant.

Inequality (1.1) has many applications related to special means-estimating error bounds for some special means and some quadrature rules and in numerical analysis etc. Hence, inequality (1.1) has attracted significant attention and interest from researchers and mathematicians. Due to this, over the years researchers have devoted much effort and time to the generalization and improvement of (1.1). Including the works in [3, 4, 24, 26], so many others are the results of these studies.

The concept of fractional calculus was developed prior to the turn of 20^{th} century. It has many applications in different fields of engineering and science including fluid flow, viscoelastic materials, rheology, diffusive transport, electrical networks, probability and electromagnetic theory. Like ordinary calculus, fractional integrals and derivatives are not defined in a unique way. Different authors have their contributions in [8, 10, 12, 23].

2. Notations and Preliminaries

Definition 2.1. [14] *Let $\breve{f} : \hat{I} \subseteq \mathbb{R} \to [0, \infty)$ be a function, then $\breve{f}$ is said to be convex (or that $\breve{f} \in Conv(\hat{I})$), provided that:*

$$\breve{f}(u\phi + (1-u)\psi) \leq u\breve{f}(\phi) + (1-u)\breve{f}(\psi),\ \forall\ \phi, \psi \in \hat{I};\ u \in [0,1].$$

Definition 2.2. [2] *Let $\breve{f} : \hat{I} \subseteq \mathbb{R} \to [0, \infty)$ be a function, then $\breve{f}$ is called $P-$function (or that $\breve{f} \in P(\hat{I})$), provided that*

$$\breve{f}(u\phi + (1-u)\psi) \leq \breve{f}(\phi) + \breve{f}(\psi),\ \forall\ \phi, \psi \in \hat{I};\ u \in [0,1].$$

Definition 2.3. [27] *Let $\breve{f} : \hat{I} \subseteq \mathbb{R} \to [0, \infty)$ be a function, then $\breve{f}$ is called $s-$Godunova-Levin function of second kind (or that $\breve{f} \in Q_s^2(\hat{I})$), provided that:*

$$\breve{f}(u\phi + (1-u)\psi) \leq \frac{\breve{f}(\phi)}{u^s} + \frac{\breve{f}(\psi)}{(1-u)^s},\ \forall\ \phi, \psi \in \hat{I};\ u \in (0,1);\ s \in [0,1].$$

Definition 2.4. [7] *Let $\breve{f} : \hat{I} \subseteq \mathbb{R} \to [0, \infty)$ be a function, then $\breve{f}$ is called $s-$convex function in the second sense (or that $f \in K_s^2(\hat{I})$), provided that:*

$$\breve{f}(u\phi + (1-u)\psi) \leq u^s\breve{f}(\phi) + (1-u)^s\breve{f}(\psi),\ \forall\ \phi, \psi \in \hat{I};\ u \in [0,1];\ s \in (0,1].$$

Definition 2.5. [30] *Let $\breve{f} : \hat{I} \subseteq \mathbb{R} \to [0, \infty)$ and $h : J \subseteq \mathbb{R} \to (0, \infty)$ be two functions such that $[0,1] \subseteq J$, then $\breve{f}$ is called $h-$convex (or that $\breve{f} \in SX(h, \hat{I})$), provided that*

$$\breve{f}(u\phi + (1-u)\psi) \leq h(u)\breve{f}(\phi) + h(1-u)\breve{f}(\psi),\ \forall\ \phi, \psi \in \hat{I}, \tag{2. 2}$$

$\breve{f}$ is said to be $h-$concave function (or that $\breve{f} \in SV(h, \hat{I})$), provided that the inequality sign in (2.2) is reversed. For different choices of the function h we get some well known inequalities. For instance,

- *for* $h \to I$, *identity function,* $SX(h, \hat{I}) = Conv(\hat{I})$.
- *for* $h(u) = u^{-s} = u^s$, $SX(h, \hat{I}) = Q_s^2(\hat{I}) = K_s^2(\hat{I})$.
- *for* $h(u) = 1$, $SX(h, \hat{I}) = P(\hat{I})$.

Definition 2.6. [23] *The right- and left-sided Riemann-Liouville fractional integrals of order* $\alpha > 0$ *of* $\breve{f}$, *denoted by* $J^{\alpha}_{\mu+}\breve{f}$ *and* $J^{\alpha}_{\nu-}\breve{f}$ *respectively, are defined as:*

$$J^{\alpha}_{\mu+}\breve{f}(\phi) = \frac{1}{\Gamma(\alpha)}\int_{\mu}^{\phi}(\phi - u)^{\alpha-1}\breve{f}(u)du, \ \phi > \mu, \tag{2. 3}$$

and

$$J^{\alpha}_{\nu-}\breve{f}(\phi) = \frac{1}{\Gamma(\alpha)}\int_{\phi}^{\nu}(u - \phi)^{\alpha-1}\breve{f}(u)du, \ \phi < \nu, \tag{2. 4}$$

where, Γ *is the Euler gamma function;* $\alpha \to 1$ *gives the classical integral and it may be noted that:*

$$J^{0}_{\mu+}\breve{f}(\phi) = J^{0}_{\nu-}\breve{f}(\phi) = \breve{f}(\phi)$$

Definition 2.7. [29] *The right- and left-sided Hadamard fractional integrals of order* $\alpha > 0$ *of* $\breve{f}$ *are defined as:*

$$H^{\alpha}_{\mu+}\breve{f}(\phi) := \frac{1}{\Gamma(\alpha)}\int_{\mu}^{\phi}\left(\ln\frac{\phi}{u}\right)^{\alpha-1}\frac{\breve{f}(u)}{u}du \tag{2. 5}$$

and

$$H^{\alpha}_{\nu-}\breve{f}(\phi) := \frac{1}{\Gamma(\alpha)}\int_{\phi}^{\nu}\left(\ln\frac{u}{\phi}\right)^{\alpha-1}\frac{\breve{f}(u)}{u}du. \tag{2. 6}$$

Definition 2.8. [13] *Let* $X^p_c(\mu, \nu)$, *for* $c \in \mathbb{R}$ *and* $p \in [1, \infty]$, *be the space of all complex valued Lebesgue measurable functions* $\breve{f}$ *for which* $\|\breve{f}\|_{X^p_c} < \infty$ *such that:*

$$\|\breve{f}\|_{X^p_c} := \sqrt[p]{\int_{\mu}^{\nu}|u^c\breve{f}(u)|^p\frac{du}{u}} \ \textit{for} \ p \in [1, \infty) \ \textit{and} \ \|\breve{f}\|_{X^{\infty}_c} := ess \sup_{\mu \leq u \leq \nu}|u^c\breve{f}(u)|.$$

Recently, Katugampola defined the following integrals unifying both Hadamard and Riemann-Liouville fractional integrals.

Definition 2.9. [12] *Let* $[\mu, \nu] \subseteq \mathbb{R}$ *be a finite interval. Then, the right- and left-sided Katugampola fractional integrals of order* $\alpha > 0$ *of* $\breve{f} \in X^p_c(\mu, \nu)$ *are defined as:*

$${}^{\varrho}L^{\alpha}_{\mu+}\breve{f}(\phi) = \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu}^{\phi}u^{\varrho-1}(\phi^{\varrho} - u^{\varrho})^{\alpha-1}\breve{f}(u)du \tag{2. 7}$$

and

$${}^{\varrho}L^{\alpha}_{\nu-}\breve{f}(\phi) = \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)}\int_{\phi}^{\nu}u^{\varrho-1}(u^{\varrho} - \phi^{\rho})^{\alpha-1}\breve{f}(u)du \tag{2. 8}$$

with $\mu < \phi < \nu$ *and* $\varrho > 0$, *provided that the integrals exist. For* $\varrho \to 1$, *integrals in* (2.7) *and* (2.8) *coincide with the integrals in* (2.3) *and* (2.4), *while for* $\varrho \to 0^+$ *integrals in* (2.7) *and* (2.8) *coincide with the integrals in* (2.5) *and* (2.6). *Katugampola fractional integral operators are well-defined on* $X^p_c(\mu, \nu)$ [11].

Definition 2.10. [30, 1] *Let $h : J \to \mathbb{R}$ be a function. Then, h is said to be super-multiplicative and super-additive, respectively, provided that the following respective inequalities hold*

$$h(\phi\psi) \geq h(\phi)h(\psi); \quad h(\phi+\psi) \geq h(\phi)+h(\psi)\ \forall\ \phi,\psi \in J.$$

3. RESULTS

Lemma 3.1. [9] *Let $\breve{f} : I \subseteq \mathbb{R} \to \mathbb{R}$ be an r-times differentiable function on I°, the interior of I, such that $\mu^\varrho, \nu^\varrho \in I^\circ$ and $\breve{f}^{(r)} \in X_c^p(\mu^\varrho, \nu^\varrho)$ for $r \in \mathbb{N}_0$. Let $\beta := \alpha - j + 1 + \frac{1-n}{\varrho}$ for $r \geq j$; $\varrho, \alpha, \beta > 0$ and $n \in \mathbb{N}$. If $\phi^\varrho \in (\mu^\varrho, \nu^\varrho)$, then*

$$\begin{aligned}
&\frac{\varrho\left(\phi^\varrho-\mu^\varrho\right)}{2}\int_0^1 u^{\alpha\varrho+r\varrho-n}\breve{f}^{(r)}(u^\varrho\phi^\varrho+\left(1-u^\varrho\right)\mu^\varrho)du-\frac{\varrho(\nu^\varrho-\phi^\varrho)}{2}\\
&\times\int_0^1 u^{\alpha\varrho+r\varrho-n}\breve{f}^{(r)}(u^\varrho\phi^\varrho+\left(1-u^\varrho\right)\nu^\varrho)du\\
&=\breve{f}^{(r-1)}(\phi^\varrho)+\sum_{\breve{k}=1}^{\breve{j}-1}\prod_{1\leq\breve{i}\leq\breve{k}}(-1)^{\breve{k}}\frac{\left[\varrho(\alpha+r+1-\breve{i})-n\right]}{2\varrho^{\breve{k}}}\\
&\times\left[\frac{(\nu^\varrho-\phi^\varrho)^{\breve{k}}+(\phi^\varrho-\mu^\varrho)^{k}}{(\phi^\varrho-\mu^\varrho)^{\breve{k}}(\nu^\varrho-\phi^\varrho)^{\breve{k}}}\right]\breve{f}^{(r-1-\breve{k})}(\phi^\varrho)\\
&+\prod_{1\leq\breve{i}\leq\breve{j}}(-1)^{\breve{j}}\frac{[\varrho(\alpha+r+1-\breve{i})-n]\Gamma(\alpha)}{2\varrho^{\breve{j}-\alpha}}\\
&\times\left[\frac{{}^{\varrho}L^{\alpha}_{\phi-}\breve{f}^{(r-\breve{j})}(\mu^\varrho)}{(\phi^\varrho-\mu^\varrho)^{\breve{j}+\beta-1}}+\frac{(-1)^{\breve{j}+1\varrho}L^{\alpha}_{\phi+}\breve{f}^{(r-\breve{j})}(\nu^\varrho)}{(\nu^\varrho-\phi^\varrho)^{\breve{j}+\beta-1}}\right]:=\Omega(\breve{f};\phi,\mu,\nu)
\end{aligned} \tag{3. 9}$$

Proof. Consider

$$I_{n,r,j} := \int_0^1 \frac{u^{\alpha\varrho+r\varrho-n-j}\breve{f}^{(r-j)}(u^\varrho\phi^\varrho+(1-u^\varrho)\mu^\varrho)du}{[\varrho u^{\varrho-1}(\phi^\varrho-\mu^\varrho)]^j}.$$

Integrating by parts yields

$$\begin{aligned}
I_{n,r,0} &= \int_0^1 \frac{u^{\alpha\varrho+r\varrho-n-0}\breve{f}^{(r-0)}(u^\varrho\phi^\varrho+(1-u^\varrho)\mu^\varrho)du}{[\varrho u^{\varrho-1}(\phi^\varrho-\mu^\varrho)]^0}\\
&= \frac{\breve{f}^{(r-1)}(\phi^\varrho)}{\varrho(\phi^\varrho-\mu^\varrho)}-(\alpha\varrho+r\varrho-n)\\
&\times\int_0^1 \frac{u^{\alpha\varrho+r\varrho-n-1}\breve{f}^{(r-1)}(u^\varrho\phi^\varrho+(1-u^\varrho)\mu^\varrho)du}{[\varrho u^{\varrho-1}(\phi^\varrho-\mu^\varrho)]^1}\\
&= \frac{\breve{f}^{(r-1)}(\phi^\varrho)}{\varrho(\phi^\varrho-\mu^\varrho)}-(\alpha\varrho+r\varrho-n)I_{n,r,1}.
\end{aligned} \tag{3. 10}$$

$$\begin{aligned}
I_{n,r,1} &= \int_0^1 \frac{u^{\alpha\varrho+r\varrho-n-1}\check{f}^{(r-1)}(u^{\varrho}\phi^{\varrho}+(1-u^{\varrho})\mu^{\varrho})du}{\varrho u^{\varrho-1}(\phi^{\varrho}-\mu^{\varrho})} \\
&= \frac{\check{f}^{(r-2)}(\phi^{\varrho})}{[\varrho(\phi^{\varrho}-\mu^{\varrho})]^2} - (\alpha\varrho+r\varrho-\varrho-n) \\
&\times \int_0^1 \frac{u^{\alpha\varrho+r\varrho-n-2}\check{f}^{(r-2)}(u^{\varrho}\phi^{\varrho}+(1-u^{\varrho})\mu^{\varrho})du}{[\varrho u^{\varrho-1}(\phi^{\varrho}-\mu^{\varrho})]^2} \\
&= \frac{\check{f}^{(r-2)}(\phi^{\varrho})}{[\varrho(\phi^{\varrho}-\mu^{\varrho})]^2} - (\alpha\varrho+r\varrho-\varrho-n)I_{n,r,2}. \qquad (3.\ 11)
\end{aligned}$$

$$\begin{aligned}
I_{n,r,2} &= \int_0^1 \frac{u^{\alpha\varrho+r\varrho-n-2}\check{f}^{(r-2)}(u^{\varrho}\phi^{\varrho}+(1-u^{\varrho})\mu^{\varrho})du}{[\varrho u^{\varrho-1}(\phi^{\varrho}-\mu^{\varrho})]^2} \\
&= \frac{\check{f}^{(r-3)}(\phi^{\varrho})}{[\varrho(\phi^{\varrho}-\mu^{\varrho})]^3} - (\alpha\varrho+r\varrho-n-2\varrho) \\
&\times \int_0^1 \frac{u^{\alpha\varrho+r\varrho-n-3}\check{f}^{(r-3)}(u^{\varrho}\phi^{\varrho}+(1-u^{\varrho})\mu^{\varrho})du}{[\varrho u^{\varrho-1}(\phi^{\varrho}-\mu^{\varrho})]^3} \\
&= \frac{\check{f}^{(r-3)}(\phi^{\varrho})}{[\varrho(\phi^{\varrho}-\mu^{\varrho})]^3} - (\alpha\varrho+r\varrho-n-2\varrho)I_{n,r,3}. \qquad (3.\ 12)
\end{aligned}$$

A combination of relations (3.10)-(3.12) yields:

$$\begin{aligned}
I_{n,r,0} &= \frac{\check{f}^{(r-1)}(\phi^{\varrho})}{\varrho(\phi^{\varrho}-\mu^{\varrho})} + \sum_{k=1}^{2}\prod_{1\le i\le k}(-1)^k\frac{[\varrho(\alpha+r+1-i)-n]}{[\varrho(\phi^{\varrho}-\mu^{\varrho})]^{k+1}}\check{f}^{(r-k-1)}(\phi^{\varrho}) \\
&+ \prod_{1\le i\le 3}(-1)^3[\rho(\alpha+r+1-i)-n]I_{n,r,3}. \qquad (3.\ 13)
\end{aligned}$$

Integrating by parts repeatedly $j-$times in relation (3.13) yields

$$\begin{aligned}
I_{n,r,0} &= \frac{\check{f}^{(r-1)}(\phi^{\varrho})}{\varrho(\phi^{\varrho}-\mu^{\varrho})} + \sum_{k=1}^{j-1}\prod_{1\le i\le k}(-1)^k\frac{[\varrho(\alpha+r+1-i)-n]}{[\varrho(\phi^{\varrho}-\mu^{\varrho})]^{k+1}}\check{f}^{(r-k-1)}(\phi^{\varrho}) \\
&+ \prod_{1\le i\le j}(-1)^j[\varrho(\alpha+r+1-i)-n]I_{n,r,j}. \qquad (3.\ 14)
\end{aligned}$$

But

$$\begin{aligned}
I_{n,r,j} &= \frac{1}{\varrho^j(\phi^{\varrho}-\mu^{\varrho})^{j+1}}\int_{\mu}^{\phi}\left(\frac{\psi^{\varrho}-\mu^{\varrho}}{\phi^{\varrho}-\mu^{\varrho}}\right)^{\frac{\varrho(\alpha+r-j-1)+1-n}{\varrho}}\psi^{\varrho-1}\check{f}^{(r-j)}(\psi^{\varrho})d\psi \\
&= \frac{{}^{\varrho}I^{\alpha}_{\phi-}\check{f}^{(r-j)}(\mu^{\varrho})\Gamma(\alpha)}{\varrho^{1-\alpha+j}(\phi^{\varrho}-\mu^{\varrho})^{j+\beta}}. \qquad (3.\ 15)
\end{aligned}$$

Combining the relations (3.14)-(3.15) implies

$$I_{n,r,0} = \frac{\check{f}^{(r-1)}(\phi^{\varrho})}{\varrho(\phi^{\varrho} - \mu^{\varrho})} + \sum_{k=1}^{j-1} \prod_{1 \le i \le k} (-1)^k \frac{[\varrho(\alpha + r + 1 - i) - n]}{[\varrho(\phi^{\varrho} - \mu^{\varrho})]^{k+1}} \check{f}^{(r-k-1)}(\phi^{\varrho})$$
$$+ \prod_{1 \le i \le j} \frac{(-1)^j [\varrho(\alpha + r + 1 - i) - n]}{\varrho^{1-\alpha+j}(\phi^{\varrho} - \mu^{\varrho})^{j+\beta}} {}^{\varrho}K^{\alpha}_{\phi-} \check{f}^{(r-m)}(\mu^{\varrho}) \Gamma(\alpha). \tag{3. 16}$$

Consider

$$J_{n,r,j} := \int_0^1 \frac{u^{\alpha\varrho + r\varrho - n - j} \check{f}^{(r-j)}(u^{\varrho}\phi^{\varrho} + (1 - u^{\varrho})\nu^{\varrho}) du}{[\varrho u^{\varrho-1}(\phi^{\varrho} - \nu^{\varrho})]^j}.$$

Similarly, using the same technique as used in (3.10)-(3.14), we have

$$J_{n,r,0} = \frac{-\check{f}^{(r-1)}(\phi^{\varrho})}{\varrho(\nu^{\varrho} - \phi^{\varrho})}$$
$$+ \sum_{k=1}^{j-1} \prod_{1 \le i \le k} \frac{(-1)^{k+1} [\varrho(\alpha + r + 1 - i) - n]}{[\varrho(\nu^{\varrho} - \phi^{\varrho})]^{k+1}} \check{f}^{(r-k-1)}(\phi^{\varrho})$$
$$+ \prod_{1 \le i \le j} (-1)^j [\varrho(\alpha + r + 1 - i) - n] J_{n,r,j}. \tag{3. 17}$$

But

$$J_{n,r,j} = \frac{1}{\varrho^j (\phi^{\varrho} - \nu^{\varrho})^{j+1}}$$
$$\times \int_{\nu}^{\phi} \left(\frac{\psi^{\varrho} - \nu^{\varrho}}{\phi^{\varrho} - \nu^{\varrho}} \right)^{\frac{\varrho(\alpha + r - j - 1) + 1 - n}{\varrho}} \psi^{\varrho - 1} \check{f}^{(r-j)}(\psi^{\varrho}) d\psi$$
$$= \frac{(-1)^j}{\varrho^j (\nu^{\varrho} - \phi^{\varrho})^{j+1}}$$
$$\times \int_{\phi}^{\nu} \left(\frac{\nu^{\varrho} - \psi^{\varrho}}{\nu^{\varrho} - \phi^{\varrho}} \right)^{\frac{\varrho(\alpha + r - j - 1) + 1 - n}{\varrho}} \psi^{\varrho - 1} \check{f}^{(r-j)}(\psi^{\varrho}) d\psi$$
$$= \frac{(-1)^j \, {}^{\varrho}K^{\alpha}_{\phi+} \check{f}^{(r-j)}(\nu^{\varrho}) \Gamma(\alpha)}{\varrho^{1-\alpha+j} (\nu^{\varrho} - \phi^{\varrho})^{j+\beta}}. \tag{3. 18}$$

Combining the relations (3.17)-(3.18), we have

$$J_{n,r,0} = \frac{-\check{f}^{(r-1)}(\phi^{\varrho})}{\varrho(\nu^{\varrho} - \phi^{\varrho})}$$
$$+ \sum_{k=1}^{j-1} \prod_{1 \le i \le k} \frac{(-1)^{k+1} [\varrho(\alpha + r + 1 - i) - n]}{[\varrho(\nu^{\varrho} - \phi^{\varrho})]^{k+1}} \check{f}^{(r-k-1)}(\phi^{\varrho})$$
$$+ \prod_{1 \le i \le j} \frac{(-1)^{2j} [\varrho(\alpha + r + 1 - i) - n]}{\varrho^{1-\alpha+j} (\nu^{\varrho} - \phi^{\varrho})^{j+\beta}} {}^{\varrho}J^{\alpha}_{\phi+} \check{f}^{(r-j)}(\nu^{\varrho}) \Gamma(\alpha). \tag{3. 19}$$

Multiplying (3.16) by $\varrho \frac{\phi^{\varrho} - \mu^{\varrho}}{2}$, (3.19) by $\varrho \frac{\nu^{\varrho} - \phi^{\varrho}}{2}$ and then subtracting the resulting equations yield the desired result (3.9). □

Remark 3.2. *By setting* $\varrho, j, n, r \to 1$, *Lemma 3.1 reduces to* [6, Lemma 2.1] *and* [25, Lemma 1].

Theorem 3.3. *Let* $\breve{f} : I \subseteq \mathbb{R} \to \mathbb{R}$ *be an* r*-times differentiable function on* I°*, the interior of* I*, such that* $\mu^{\varrho}, \nu^{\varrho} \in I^{\circ}$ *and* $\breve{f}^{(r)} \in X_c^p(\mu^{\varrho}, \nu^{\varrho})$ *for* $r \in \mathbb{N}_0$*. Let* $\beta := \alpha - j + 1 + \frac{1-n}{\varrho}$ *for* $r \geq j$*;* $\varrho, \alpha, \beta > 0$ *and* $n \in \mathbb{N}$*. Moreover, if* $\phi^{\varrho} \in (\mu^{\varrho}, \nu^{\varrho})$ *and* $\left|\breve{f}^{(r)}\right| \in SX(h, I_\eta^{\varrho})$ *is such that* $\left|\breve{f}^{(r)}(\phi^{\varrho})\right| \leq M$ *for* $M > 0$*, then*

$$\left|\Omega(\breve{f}; \phi, \mu, \nu)\right| \leq M\varrho\frac{\nu^{\varrho} - \mu^{\varrho}}{2} \int_0^1 u^{\alpha\varrho + r\varrho - n} \left\{h(u^{\varrho}) + h(1 - u^{\varrho})\right\} du.$$

Proof. Applications of Lemma 3.1, boundedness and $h-$convexity of $\left|\breve{f}^{(r)}\right|$ yield:

$$\begin{aligned}
\left|\Omega(\breve{f}; \phi, \mu, \nu)\right| &\leq \varrho\frac{\phi^{\varrho} - \mu^{\varrho}}{2} \int_0^1 u^{\alpha\varrho + r\varrho - n} \left|\breve{f}^{(r)}(u^{\varrho}\phi^{\varrho} + (1 - u^{\varrho})\mu^{\varrho})\right| du \\
&+ \varrho\frac{\nu^{\varrho} - \phi^{\varrho}}{2} \int_0^1 u^{\alpha\varrho + r\varrho - n} \left|\breve{f}^{(r)}(u^{\varrho}\phi^{\varrho} + (1 - u^{\varrho})\nu^{\varrho})\right| du \\
&\leq \varrho\frac{\phi^{\varrho} - \mu^{\varrho}}{2} \int_0^1 u^{\alpha\varrho + r\varrho - n} \left\{h(u^{\varrho}) \left|\breve{f}^{(r)}(\phi^{\varrho})\right| + h(1 - u^{\varrho}) \left|\breve{f}^{(r)}(\mu^{\varrho})\right|\right\} du \\
&+ \varrho\frac{\nu^{\varrho} - \phi^{\varrho}}{2} \int_0^1 u^{\alpha\varrho + r\varrho - n} \left\{h(u^{\varrho}) \left|\breve{f}^{(r)}(\phi^{\varrho})\right| + h(1 - u^{\varrho}) \left|\breve{f}^{(r)}(\nu^{\varrho})\right|\right\} du \\
&\leq M\varrho\frac{\phi^{\varrho} - \mu^{\varrho}}{2} \int_0^1 u^{\alpha\varrho + r\varrho - n} \left\{h(u^{\varrho}) + h(1 - u^{\varrho})\right\} du \\
&+ M\varrho\frac{\nu^{\varrho} - \phi^{\varrho}}{2} \int_0^1 u^{\alpha\varrho + r\varrho - n} \left\{h(u^{\varrho}) + h(1 - u^{\varrho})\right\} du \\
&= M\varrho\frac{\nu^{\varrho} - \mu^{\varrho}}{2} \int_0^1 u^{\alpha\varrho + r\varrho - n} \left\{h(u^{\varrho}) + h(1 - u^{\varrho})\right\} du.
\end{aligned}$$

Hence, the proof has been completed. □

Corollary 3.4. *Let the conditions of Theorem 3.3 be satisfied for* $h \in P(I_\eta^{\varrho})$*, then*

$$\left|\Omega(\breve{f}; \phi, \mu, \nu)\right| \leq M\varrho\frac{\nu^{\varrho} - \mu^{\varrho}}{\alpha\varrho + r\varrho - n + 1}.$$

Corollary 3.5. *Let the conditions of Theorem 3.3 be satisfied for* $j, r, n, \varrho \to 1$*, if* h *is super-multiplicative and* $h(u) \geq u$*, then*

$$\begin{aligned}
&\left|\breve{f}(\phi) - \Gamma(\alpha + 1)\left[\frac{J_{\phi-}^{\alpha}\breve{f}(\mu)}{2(\phi - \mu)^{\alpha}} + \frac{J_{\phi+}^{\alpha}\breve{f}(\nu)}{2(\nu - \phi)^{\alpha}}\right]\right| \\
&\leq M\frac{(\nu - \mu)}{2} \int_0^1 \left[h(u^{\alpha+1}) + h(u^{\alpha}(1 - u))\right] du.
\end{aligned}$$

Remark 3.6.
- *On letting* $r, n, j \to 1$*; Theorem 3.3 and Corollary 3.4 refine as well as coincide with* [6, Theorem 2.2] *and* [6, Corollary 2.3]*, respectively.*

- *On letting* $\varrho, r, n, j \to 1$, *Theorem 3.3 refines as well as coincides with* [25, Theorem 1] *and in particular for* $\phi \to \frac{\mu+\nu}{2}$ *and* $\alpha \to 1$ *it refines and coincides with* [25, Corollary 3].
- *On letting* $\varrho, r, n, j \to 1$, *and* $h \in Conv([\mu,\nu])$, *Theorem 3.3 refines and coincides with* [25, Corollary 1]. *Moreover, for* $h \in K_s^2([\mu,\nu])$, *Theorem 3.3 refines and coincides with* [25, Corollary 2].

Theorem 3.7. *Let the conditions of Theorem 3.3 be satisfied. Moreover, if* $\breve{p}, \breve{q} > 1$ *for which* $\breve{p} = \frac{\breve{q}}{\breve{q}-1}$ *and* $\left|\breve{f}^{(r)}\right|^{\breve{q}} \in SX(h, I_\eta^\varrho)$, *then*

$$\left|\Omega(\breve{f};\phi,\mu,\nu)\right| \leq \varrho\frac{\nu^\varrho-\mu^\varrho}{2\breve{p}(\breve{p}\alpha\varrho+\breve{p}r\varrho-n\breve{p}+1)} + M\varrho\frac{\nu^\varrho-\mu^\varrho}{2\breve{q}}\int_0^1 \{h(u^\varrho)+h(1-u^\varrho)\}\,du.$$

Proof. Applications of Lemma 3.1, Young's inequality in [5, Page 10] and $|\breve{f}^{(r)}|^{\breve{q}} \in SX(h, I_\eta^\varrho)$ yield:

$$\begin{aligned}
&\left|\Omega(\breve{f};\phi,\mu,\nu)\right| \leq \varrho\frac{\phi^\varrho-\mu^\varrho}{2}\int_0^1 u^{\alpha\varrho+r\varrho-n}\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\mu^\varrho)\right|du\\
&+\varrho\frac{\nu^\varrho-\phi^\varrho}{2}\int_0^1 u^{\alpha\varrho+r\varrho-n}\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\nu^\varrho)\right|du\\
&\leq \varrho\frac{\phi^\varrho-\mu^\varrho}{2}\left\{\frac{1}{\breve{p}}\int_0^1\left(u^{\alpha\varrho+r\varrho-n}\right)^{\breve{p}}du+\frac{1}{\breve{q}}\int_0^1\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\mu^\varrho)\right|^{\breve{q}}dt\right\}\\
&+\varrho\frac{\nu^\varrho-\phi^\varrho}{2}\left\{\frac{1}{\breve{p}}\int_0^1\left(u^{\alpha\varrho+r\varrho-n}\right)^{\breve{p}}du+\frac{1}{\breve{q}}\int_0^1\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\nu^\varrho)\right|^{\breve{q}}du\right\}\\
&=\varrho\frac{\nu^\varrho-\mu^\varrho}{2\breve{p}(\breve{p}\alpha\varrho+\breve{p}r\varrho-n\breve{p}+1)}+\varrho\frac{\phi^\varrho-\mu^\varrho}{2\breve{q}}\int_0^1\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\mu^\varrho)\right|^{\breve{q}}du\\
&+\varrho\frac{\nu^\varrho-\phi^\varrho}{2\breve{q}}\int_0^1\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\nu^\varrho)\right|^{\breve{q}}du\\
&\leq\varrho\frac{\nu^\varrho-\mu^\varrho}{2\breve{p}(\breve{p}\alpha\varrho+\breve{p}r\varrho-n\breve{p}+1)}\\
&+\varrho\frac{\phi^\varrho-\mu^\varrho}{2\breve{q}}\int_0^1\left\{h(u^\varrho)\left|\breve{f}^{(r)}(\phi^\varrho)\right|^{\breve{q}}+h(1-u^\varrho)\left|\breve{f}^{(r)}(\mu^\varrho)\right|^{\breve{q}}\right\}du\\
&+\varrho\frac{\nu^\varrho-\phi^\varrho}{2\breve{q}}\int_0^1\left\{h(u^\varrho)\left|\breve{f}^{(r)}(\phi^\varrho)\right|^{\breve{q}}+h(1-u^\varrho)\left|\breve{f}^{(r)}(\nu^\varrho)\right|^{\breve{q}}\right\}du\\
&\leq\varrho\frac{\nu^\varrho-\mu^\varrho}{2\breve{p}(\breve{p}\alpha\rho+\breve{p}r\rho-n\breve{p}+1)}+M\varrho\frac{\phi^\varrho-\mu^\varrho}{2\breve{q}}\int_0^1\{h(u^\varrho)+h(1-u^\varrho)\}du\\
&+M\varrho\frac{\nu^\varrho-\phi^\varrho}{2\breve{q}}\int_0^1\{h(u^\varrho)+h(1-u^\varrho)\}du\\
&=\varrho\frac{\nu^\varrho-\mu^\varrho}{2\breve{p}(\breve{p}\alpha\varrho+\breve{p}r\varrho-n\breve{p}+1)}+M\varrho\frac{\nu^\varrho-\mu^\varrho}{2\breve{q}}\int_0^1\{h(u^\varrho)+h(1-u^\varrho)\}du.
\end{aligned}$$

Hence the proof has been completed. □

Corollary 3.8. *Let the conditions of Theorem 3.7 be satisfied for* $h \in P(I_\eta^\varrho)$*, then*

$$\left|\Omega(\breve{f};\phi,\mu,\nu)\right| \le \frac{\varrho\left\{\breve{q}+2M\breve{p}(\breve{p}\alpha\varrho+\breve{p}r\varrho-n\breve{p}+1)\right\}\left\{\nu^\varrho-\mu^\varrho\right\}}{2\breve{p}\breve{q}(\breve{p}\alpha\varrho+\breve{p}r\varrho-n\breve{p}+1)}.$$

Corollary 3.9. *Let the conditions of Theorem 3.7 be satisfied for* $j, r, n, \varrho \to 1$*, if* h *is super-additive and* $h(u) \ge u$*, then*

$$\left|\breve{f}(\phi)-\Gamma(\alpha+1)\left[\frac{J_{\phi-}^\alpha\breve{f}(\mu)}{2(\phi-\mu)^\alpha}+\frac{J_{\phi+}^\alpha\breve{f}(\nu)}{2(\nu-\phi)^\alpha}\right]\right| \le \frac{\nu-\mu\{\breve{q}+M\breve{p}(\breve{p}\alpha+1)h(1)\}}{2\breve{p}\breve{q}(\breve{p}\alpha+1)}.$$

Theorem 3.10. *Let the conditions of Theorem 3.7 be satisfied, then*

$$\begin{aligned}\left|\Omega(\breve{f};\phi,\mu,\nu)\right| &\le M\varrho\frac{\nu^\varrho-\mu^\varrho}{2}\left[\frac{\sqrt[\breve{q}]{\int_0^1 u^\varrho\left\{h(u^\varrho)+h(1-u^\varrho)\right\}du}}{\sqrt[\breve{p}]{\alpha\varrho\breve{p}+r\breve{p}\varrho+\varrho-n\breve{p}+1}}\right.\\ &\left.+\frac{\sqrt[\breve{p}]{\varrho}\sqrt[\breve{q}]{\int_0^1(1-u^\varrho)\left\{h(u^\varrho)+h(1-u^\varrho)\right\}du}}{\sqrt[\breve{p}]{(\alpha\varrho\breve{p}+r\breve{p}\varrho-n\breve{p}+1)(\alpha\varrho\breve{p}+r\breve{p}\varrho+\varrho-n\breve{p}+1)}}\right]\\ &\le M\varrho\frac{\nu^\varrho-\mu^\varrho}{\sqrt[\breve{p}]{\alpha\varrho\breve{p}+r\varrho\breve{p}-n\breve{p}+1}}\sqrt[\breve{q}]{2^{1-\breve{q}}\int_0^1 h(u^\varrho)du}.\end{aligned}$$

Proof. Applications of Lemma 3.1 and Hölder-İşcan integral inequality in [28, Theorem 1.4] yield:

$$\begin{aligned}&\left|\Omega(\breve{f};\phi,\mu,\nu)\right| \le \varrho\frac{\phi^\varrho-\mu^\varrho}{2}\int_0^1 u^{\alpha\varrho+r\varrho-n}\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\mu^\varrho)\right|du\\ &+\varrho\frac{\nu^\varrho-\phi^\varrho}{2}\int_0^1 u^{\alpha\varrho+r\varrho-n}\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\nu^\varrho)\right|du\\ &\le \varrho\frac{\phi^\varrho-\mu^\varrho}{2}\\ &\times\left[\sqrt[\breve{p}]{\int_0^1(1-u^\varrho)\left|u^{\alpha\varrho+r\varrho-n}\right|^{\breve{p}}du}\sqrt[\breve{q}]{\int_0^1(1-u^\varrho)\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\mu^\varrho)\right|^{\breve{q}}du}\right.\\ &\left.+\sqrt[\breve{p}]{\int_0^1 u^\varrho\left|u^{\alpha\varrho+r\varrho-n}\right|^{\breve{p}}du}\sqrt[\breve{q}]{\int_0^1 u^\varrho\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\mu^\varrho)\right|^{\breve{q}}du}\right]\\ &+\varrho\frac{\nu^\varrho-\phi^\varrho}{2}\left[\sqrt[\breve{p}]{\int_0^1(1-u^\varrho)\left|u^{\alpha\varrho+r\varrho-n}\right|^{\breve{p}}du}\right.\\ &\times\sqrt[\breve{q}]{\int_0^1(1-u^\varrho)\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\nu^\varrho)\right|^{\breve{q}}du}\\ &\left.+\sqrt[\breve{p}]{\int_0^1 u^\varrho\left|u^{\alpha\varrho+r\varrho-n}\right|^{\breve{p}}du}\sqrt[\breve{q}]{\int_0^1 u^\varrho\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\nu^\varrho)\right|^{\breve{q}}du}\right]\end{aligned}$$

$$\leq \varrho\frac{\phi^{\varrho}-\mu^{\varrho}}{2}\left[\frac{\sqrt[\breve{p}]{\varrho}\sqrt[\breve{q}]{\int_0^1(1-u^{\varrho})\left\{h(u^{\varrho})\left|\breve{f}^{(r)}(\phi^{\varrho})\right|^{\breve{q}}+h(1-u^{\varrho})\left|\breve{f}^{(r)}(\mu^{\varrho})\right|^{\breve{q}}\right\}du}}{\sqrt[\breve{p}]{(\alpha\varrho\breve{p}+r\breve{p}\varrho-n\breve{p}+1)(\alpha\varrho\breve{p}+r\breve{p}\varrho+\varrho-n\breve{p}+1)}}\right.$$

$$\left.+\frac{\sqrt[\breve{q}]{\int_0^1u^{\varrho}\left\{h(u^{\varrho})\left|\breve{f}^{(r)}(\phi^{\varrho})\right|^{\breve{q}}+h(1-u^{\varrho})\left|\breve{f}^{(r)}(\mu^{\varrho})\right|^{\breve{q}}\right\}du}}{\sqrt[\breve{p}]{\alpha\varrho\breve{p}+r\breve{p}\varrho+\rho-n\breve{p}+1}}\right]$$

$$+\varrho\frac{\nu^{\varrho}-\phi^{\varrho}}{2}$$

$$\times\left[\frac{\sqrt[\breve{p}]{\varrho}\sqrt[\breve{q}]{\int_0^1(1-u^{\varrho})\int_0^1\left\{h(u^{\varrho})\left|\breve{f}^{(r)}(\phi^{\varrho})\right|^{\breve{q}}+h(1-u^{\varrho})\left|\breve{f}^{(r)}(\nu^{\varrho})\right|^{\breve{q}}\right\}du}}{\sqrt[\breve{p}]{(\alpha\varrho\breve{p}+r\breve{p}\varrho-n\breve{p}+1)(\alpha\varrho\breve{p}+r\breve{p}\varrho+\varrho-n\breve{p}+1)}}\right.$$

$$\left.+\frac{\sqrt[\breve{q}]{\int_0^1u^{\varrho}\int_0^1\left\{h(u^{\varrho})\left|\breve{f}^{(r)}(\phi^{\varrho})\right|^{\breve{q}}+h(1-u^{\varrho})\left|\breve{f}^{(r)}(\nu^{\varrho})\right|^{\breve{q}}\right\}du}}{\sqrt[\breve{p}]{\alpha\varrho\breve{p}+r\breve{p}\varrho+\varrho-n\breve{p}+1}}\right]$$

$$\leq M\varrho\frac{\phi^{\varrho}-\mu^{\varrho}}{2}\left[\frac{\sqrt[\breve{p}]{\varrho}\sqrt[\breve{q}]{\int_0^1(1-u^{\varrho})\{h(u^{\varrho})+h(1-u^{\varrho})\}du}}{\sqrt[\breve{p}]{(\alpha\varrho\breve{p}+r\breve{p}\varrho-n\breve{p}+1)(\alpha\varrho\breve{p}+r\breve{p}\varrho+\varrho-n\breve{p}+1)}}\right.$$

$$\left.+\frac{\sqrt[\breve{q}]{\int_0^1u^{\varrho}\{h(u^{\varrho})+h(1-u^{\varrho})\}du}}{\sqrt[\breve{p}]{(\alpha\varrho\breve{p}+r\breve{p}\varrho+\varrho-n\breve{p}+1)}}\right]+M\varrho\frac{\nu^{\varrho}-\phi^{\varrho}}{2}$$

$$\times\left[\frac{\sqrt[\breve{p}]{\varrho}\sqrt[\breve{q}]{\int_0^1(1-u^{\varrho})\{h(u^{\varrho})+h(1-u^{\varrho})\}du}}{\sqrt[\breve{p}]{(\alpha\varrho\breve{p}+r\breve{p}\varrho-n\breve{p}+1)(\alpha\varrho\breve{p}+r\breve{p}\varrho+\varrho-n\breve{p}+1)}}\right.$$

$$\left.+\frac{\sqrt[\breve{q}]{\int_0^1u^{\varrho}\{h(u^{\varrho})+h(1-u^{\varrho})\}du}}{\sqrt[\breve{p}]{(\alpha\varrho\breve{p}+r\breve{p}\varrho+\varrho-n\breve{p}+1)}}\right]$$

$$=M\varrho\frac{\nu^{\varrho}-\mu^{\varrho}}{2}\left[\frac{\sqrt[\breve{p}]{\varrho}\sqrt[\breve{q}]{\int_0^1(1-u^{\varrho})\{h(u^{\varrho})+h(1-u^{\varrho})\}du}}{\sqrt[\breve{p}]{(\alpha\varrho\breve{p}+r\breve{p}\varrho-n\breve{p}+1)(\alpha\varrho\breve{p}+r\breve{p}\varrho+\varrho-n\breve{p}+1)}}\right.$$

$$\left.+\frac{\sqrt[\breve{q}]{\int_0^1u^{\varrho}\{h(u^{\varrho})+h(1-u^{\varrho})\}du}}{\sqrt[\breve{p}]{(\alpha\varrho\breve{p}+r\breve{p}\varrho+\varrho-n\breve{p}+1)}}\right]$$

$$\leq M\varrho\frac{\nu^{\varrho}-\mu^{\varrho}}{2\sqrt[\breve{p}]{\alpha\varrho\breve{p}+r\varrho\breve{p}-n\breve{p}+1}}\sqrt[\breve{q}]{\int_0^1\{h(u^{\varrho})+h(1-u^{\varrho})\}du}$$

$$=M\varrho\frac{\nu^{\varrho}-\mu^{\varrho}}{2\sqrt[\breve{p}]{\alpha\varrho\breve{p}+r\varrho\breve{p}-n\breve{p}+1}}\sqrt[\breve{q}]{2\int_0^1h(u^{\varrho})du}$$

$$=M\varrho\frac{\nu^{\varrho}-\mu^{\varrho}}{\sqrt[\breve{p}]{\alpha\varrho\breve{p}+r\varrho\breve{p}-n\breve{p}+1}}\sqrt[\breve{q}]{2^{1-\breve{q}}\int_0^1h(u^{\varrho})du}$$

Hence, the proof has been completed. □

Corollary 3.11. *Let the conditions of Theorem 3.7 be satisfied for $h \in P(I_\eta^\varrho)$, then*

$$\begin{aligned}\left|\Omega(\breve{f};\phi,\mu,\nu)\right| &\leq \frac{\nu^\varrho-\mu^\varrho}{2}\frac{M\varrho\sqrt[\breve{q}]{\frac{2}{\varrho+1}}\left\{\varrho+\sqrt[\breve{p}]{\alpha\varrho\breve{p}+r\breve{p}\varrho-n\breve{p}+1}\right\}}{\sqrt[\breve{p}]{(\alpha\varrho\breve{p}+r\breve{p}\varrho-n\breve{p}+1)(\alpha\varrho\breve{p}+r\breve{p}\varrho+\varrho-n\breve{p}+1)}}\\ &\leq \sqrt[\breve{q}]{2^{1-\breve{q}}}M\varrho\frac{\nu^\varrho-\mu^\varrho}{\sqrt[\breve{p}]{\alpha\varrho\breve{p}+r\varrho\breve{p}-n\breve{p}+1}}.\end{aligned}$$

Remark 3.12.
- *On letting $r,n,j \to 1$, Theorem 3.7 and Corollary 3.8 refine as well as coincide with* [6, Theorem 2.5] *and* [6, Corollary 2.6], *respectively.*
- *On letting $\varrho,r,n,j \to 1$, Theorem 3.7 refines as well as coincides with* [25, Theorem 2] *and in particular for $\phi \to \frac{\mu+\nu}{2}$ it refines and coincides with* [25, Corollary 6].
- *On letting $\varrho,r,n,j \to 1$ and $h \in Conv([\mu,\nu])$, Theorem 3.7 refines and coincides with* [25, Corollary 4]. *Moreover, for $h \in K_s^2([\mu,\nu])$ Theorem 3.7 refines and coincides with* [25, Corollary 5].

Theorem 3.13. *Let the conditions of Theorem 3.10 be satisfied; Moreover, if $\breve{q} \geq 1$ for which $\left|\breve{f}^{(r)}\right|^{\breve{q}} \in SX(h, I_\eta^\varrho)$, then*

$$\begin{aligned}\left|\Omega(\breve{f};\phi,\mu,\nu)\right| &\leq M\varrho\frac{\nu^\varrho-\mu^\varrho}{2}\left[\frac{\sqrt[\breve{q}]{\varrho^{\breve{q}-1}\int_0^1(1-u^\varrho)u^{\alpha\varrho+r\varrho-n}\left\{h(u^\varrho)+h(1-u^\varrho)\right\}du}}{\sqrt[\breve{q}]{[(\alpha\varrho+r\varrho-n+\varrho+1)(\alpha\varrho+r\varrho-n+1)]^{\breve{q}-1}}}\right.\\ &\left.+\frac{\sqrt[\breve{q}]{\int_0^1 u^{\alpha\varrho+r\varrho+\varrho-n}\left\{h(u^\varrho)+h(1-u^\varrho)\right\}du}}{\sqrt[\breve{q}]{(\alpha\varrho+r\varrho-n+\varrho+1)^{\breve{q}-1}}}\right]\\ &\leq M\varrho\frac{\nu^\varrho-\mu^\varrho}{2\sqrt[\breve{q}]{(\alpha\varrho+r\varrho-n+1)^{\breve{q}-1}}}\sqrt[\breve{q}]{\int_0^1 u^{\alpha\varrho+r\varrho-n}\left\{h(u^\varrho)+h(1-u^\varrho)\right\}du}.\end{aligned}$$

Proof. Applications of Lemma 3.1 and improved power mean inequality in [28, Theorem 1.5] yield:

$$\begin{aligned}\left|\Omega(\breve{f};\phi,\mu,\nu)\right| &\leq \varrho\frac{\phi^\varrho-\mu^\varrho}{2}\int_0^1 u^{\alpha\varrho+r\varrho-n}\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\mu^\varrho)\right|du\\ &+\varrho\frac{\nu^\varrho-\phi^\varrho}{2}\times\int_0^1 u^{\alpha\varrho+r\varrho-n}\left|\breve{f}^{(r)}(u^\varrho\phi^\varrho+(1-u^\varrho)\nu^\varrho)\right|du\end{aligned}$$

$$\leq \varrho \frac{\phi^{\varrho} - \mu^{\varrho}}{2} \left[\sqrt[\breve{q}]{\left(\int_0^1 (1-u^{\varrho}) \left| u^{\alpha\varrho + r\varrho - n} \right| du \right)^{\breve{q}-1}} \right.$$
$$\times \sqrt[\breve{q}]{\int_0^1 (1-u^{\varrho}) \left| u^{\alpha\varrho + r\varrho - n} \right| \left| \breve{f}^{(r)}(u^{\varrho}\phi^{\varrho} + (1-u^{\varrho})\mu^{\varrho}) \right|^{\breve{q}} du}$$
$$\left. + \sqrt[\breve{q}]{\left(\int_0^1 u^{\varrho} \left| u^{\alpha\varrho + r\varrho - n} \right| du \right)^{\breve{q}-1}} \sqrt[\breve{q}]{\int_0^1 u^{\varrho} \left| u^{\alpha\varrho + r\varrho - n} \right| \left| \breve{f}^{(r)}(u^{\varrho}\phi^{\varrho} + (1-u^{\varrho})\mu^{\varrho}) \right|^{\breve{q}} dt} \right]$$
$$+ \varrho \frac{\nu^{\varrho} - \phi^{\varrho}}{2} \left[\sqrt[\breve{q}]{\left(\int_0^1 (1-u^{\varrho}) \left| u^{\alpha\varrho + r\varrho - n} \right| du \right)^{\breve{q}-1}} \right.$$
$$\times \sqrt[\breve{q}]{\int_0^1 (1-u^{\varrho}) \left| u^{\alpha\varrho + r\varrho - n} \right| \left| \breve{f}^{(r)}(u^{\varrho}\phi^{\varrho} + (1-u^{\varrho})\nu^{\varrho}) \right|^{\breve{q}} du}$$
$$\left. + \sqrt[\breve{q}]{\left(\int_0^1 u^{\varrho} \left| u^{\alpha\varrho + r\varrho - n} \right| du \right)^{\breve{q}-1}} \sqrt[\breve{q}]{\int_0^1 u^{\varrho} \left| u^{\alpha\varrho + r\varrho - n} \right| \left| \breve{f}^{(r)}(u^{\varrho}\phi^{\varrho} + (1-u^{\varrho})\nu^{\varrho}) \right|^{\breve{q}} du} \right]$$
$$\leq \varrho \frac{\phi^{\varrho} - \mu^{\varrho}}{2}$$
$$\times \left[\frac{\sqrt[\breve{q}]{\varrho^{\breve{q}-1} \int_0^1 (1-u^{\varrho}) u^{\alpha\varrho + r\varrho - n} \left\{ h(u^{\varrho}) \left| \breve{f}^{(r)}(\phi^{\varrho}) \right|^{\breve{q}} + h(1-u^{\varrho}) \left| \breve{f}^{(r)}(\mu^{\varrho}) \right|^{\breve{q}} \right\} du}}{\sqrt[\breve{q}]{[(\alpha\varrho + r\varrho - n + \varrho + 1)(\alpha\varrho + r\varrho - n + 1)]^{\breve{q}-1}}} \right.$$
$$\left. + \frac{\sqrt[\breve{q}]{\int_0^1 u^{\alpha\varrho + r\varrho + \varrho - n} \int_0^1 \left\{ h(u^{\varrho}) \left| \breve{f}^{(r)}(\phi^{\varrho}) \right|^{\breve{q}} + h(1-u^{\varrho}) \left| \breve{f}^{(r)}(\mu^{\varrho}) \right|^{\breve{q}} \right\} du}}{\sqrt[\breve{q}]{(\alpha\varrho + r\varrho - n + \varrho + 1)^{\breve{q}-1}}} \right]$$
$$+ \varrho \frac{\nu^{\varrho} - \phi^{\varrho}}{2}$$
$$\times \left[\frac{\sqrt[\breve{q}]{\varrho^{\breve{q}-1} \int_0^1 (1-u^{\varrho}) u^{\alpha\varrho + r\varrho - n} \left\{ h(u^{\varrho}) \left| \breve{f}^{(r)}(\phi^{\varrho}) \right|^{\breve{q}} + h(1-u^{\varrho}) \left| \breve{f}^{(r)}(\nu^{\varrho}) \right|^{\breve{q}} \right\} du}}{\sqrt[\breve{q}]{[(\alpha\varrho + r\varrho - n + \varrho + 1)(\alpha\varrho + r\varrho - n + 1)]^{\breve{q}-1}}} \right.$$
$$\left. + \frac{\sqrt[\breve{q}]{\int_0^1 u^{\alpha\varrho + r\varrho + \varrho - n} \left\{ h(u^{\varrho}) \left| \breve{f}^{(r)}(\phi^{\varrho}) \right|^{\breve{q}} + h(1-u^{\varrho}) \left| \breve{f}^{(r)}(\nu^{\varrho}) \right|^{\breve{q}} \right\} du}}{\sqrt[\breve{q}]{(\alpha\varrho + r\varrho - n + \varrho + 1)^{\breve{q}-1}}} \right]$$
$$\leq M \varrho \frac{\phi^{\varrho} - \mu^{\varrho}}{2} \left[\frac{\sqrt[\breve{q}]{\varrho^{\breve{q}-1} \int_0^1 (1-u^{\varrho}) u^{\alpha\varrho + r\varrho - n} \left\{ h(u^{\varrho}) + h(1-u^{\varrho}) \right\} du}}{\sqrt[\breve{q}]{[(\alpha\varrho + r\varrho - n + \varrho + 1)(\alpha\varrho + r\varrho - n + 1)]^{\breve{q}-1}}} \right.$$
$$\left. + \frac{\sqrt[\breve{q}]{\int_0^1 u^{\alpha\varrho + r\varrho + \varrho - n} \left\{ h(u^{\varrho}) + h(1-u^{\varrho}) \right\} du}}{\sqrt[\breve{q}]{(\alpha\varrho + r\varrho - n + \varrho + 1)^{\breve{q}-1}}} \right]$$

$$+ M\varrho\frac{\nu^\varrho-\phi^\varrho}{2}\left[\frac{\sqrt[\breve{q}]{\varrho^{\breve{q}-1}\int_0^1(1-u^\varrho)u^{\alpha\varrho+r\varrho-n}\left\{h(u^\varrho)+h(1-u^\varrho)\right\}du}}{\sqrt[\breve{q}]{[(\alpha\varrho+r\varrho-n+\varrho+1)(\alpha\varrho+r\varrho-n+1)]^{\breve{q}-1}}}\right.$$
$$\left.+\frac{\sqrt[\breve{q}]{\int_0^1 u^{\alpha\varrho+r\varrho+\varrho-n}\left\{h(u^\varrho)+h(1-u^\varrho)\right\}du}}{\sqrt[\breve{q}]{(\alpha\varrho+r\varrho-n+\varrho+1)^{\breve{q}-1}}}\right]$$
$$\leq M\varrho\frac{\nu^\varrho-\mu^\varrho}{2}\left[\frac{\sqrt[\breve{q}]{\varrho^{\breve{q}-1}\int_0^1(1-u^\varrho)u^{\alpha\varrho+r\varrho-n}\left\{h(u^\varrho)+h(1-u^\varrho)\right\}du}}{\sqrt[\breve{q}]{[(\alpha\varrho+r\varrho-n+\varrho+1)(\alpha\varrho+r\varrho-n+1)]^{\breve{q}-1}}}\right.$$
$$\left.+\frac{\sqrt[\breve{q}]{\int_0^1 u^{\alpha\varrho+r\varrho+\varrho-n}\left\{h(u^\varrho)+h(1-u^\varrho)\right\}du}}{\sqrt[\breve{q}]{(\alpha\varrho+r\varrho-n+\varrho+1)^{\breve{q}-1}}}\right]$$
$$\leq M\varrho\frac{\nu^\varrho-\mu^\varrho}{2\sqrt[\breve{q}]{(\alpha\varrho+r\varrho-n+1)^{\breve{q}-1}}}\sqrt[\breve{q}]{\int_0^1 u^{\alpha\varrho+r\varrho-n}\left\{h(u^\varrho)+h(1-u^\varrho)\right\}du}$$

Hence, the proof has been completed. □

Corollary 3.14. *Let the conditions of Theorem 3.13 be satisfied for* $h\in P(I_\eta^\varrho)$, *then*

$$\left|\Omega(\breve{f};\phi,\mu,\nu)\right|$$
$$\leq M\varrho\frac{\nu^\varrho-\mu^\varrho}{\sqrt[\breve{q}]{2^{\breve{q}-1}}}\left[\frac{\alpha\varrho+r\varrho-n+2\varrho+1}{(\alpha\varrho+r\varrho-n+\varrho+1)(\alpha\varrho+r\varrho-n+1)}\right]$$
$$\leq \sqrt[\breve{q}]{2^{\breve{q}-1}}M\varrho\frac{\nu^\varrho-\mu^\varrho}{(\alpha\varrho+r\varrho-n+1)}$$

Remark 3.15.

- *On letting* $r,n,j\to 1$, *Theorem 3.13 and Corollary 3.14 refine as well as coincide with* [6, Theorem 2.8] *and* [6, Corollary 2.10], *respectively.*
- *On letting* $\varrho,r,n,j\to 1$, *Theorem 3.13 refines as well as coincides with* [25, Theorem 3] *and in particular for* $\phi\to\frac{\mu+\nu}{2}$ *it refines and coincides with* [25, Corollary 9].
- *On letting* $\varrho,r,n,j\to 1$ *and* $h\in Conv([\mu,\nu])$, *Theorem 3.13 refines and coincides with* [25, Corollary 7]. *Moreover, for* $h\in K_s^2([\mu,\nu])$, *Theorem 3.13 refines and coincides with* [25, Corollary 8].

4. Applications related to special means

For arbitrary positive numbers $\mu,\nu(\mu\neq\nu)$, we consider the means as given;

1: Arithmetic mean:

$$A(\mu,\nu)=\frac{\mu+\nu}{2}.$$

2: Generalized logarithmic mean:

$$L_{\breve{p}}(\mu,\nu)=\left[\frac{\nu^{\breve{p}+1}-\mu^{\breve{p}+1}}{(\breve{p}+1)(\nu-\mu)}\right],\quad \breve{p}\in\mathbb{R}\backslash\{-1,0\}.$$

Now, utilize the outcome of Section 3, we describe few applications related to special means concern with real numbers. In [30], the related example is describe below:

Example 4.1. [30]
Consider the function h defined as $h(\phi) = (c+\phi)^{\breve{p}-1},\ \phi \geq 0.$

1: *In case $c = 0$, the function h will be multiplicative function.*
2: *In case $c \geq 1$, if $\breve{p} \in (0,1)$ the function h will be supermultiplicative where as if $\breve{p} > 1$ the function will be a sub-multiplicative.*
3: *In case $c = 1$ and $\breve{p} \in (0,1)$, with expression $h(u) = (1+u)^{\breve{p}-1},\ u \geq 0$ will be a supermultiplicative.*

Let $\breve{f}(\phi) = \phi^n,\ \phi > 0, |n| \geq 2$ is $h-$convex functions.

Proposition 4.2. *Let $0 < \mu < \nu, \breve{p} \in (0,1)$. Then*

$$|A^n(\mu,\nu) - L_n^n(\mu,\nu)| \leq \frac{M(\nu-\mu)}{2}\left[\frac{2^{\breve{p}}-1}{\breve{p}}\right]$$

Proof. We can derive inequality from Theorem 3.3 with $\varrho, r, \alpha, n, j \to 1$, and $\phi \to \frac{\mu+\nu}{2}$ which is applied for the $h-$convex functions that are $\breve{f} : \mathbb{R} \to \mathbb{R}, \breve{f}(\phi) = \phi^n$, and $h : \mathbb{R} \to \mathbb{R},\ h(u) = (1+u)^{\breve{p}-1}$ for $\breve{p} \in (0,1)$. Further explanations can be neglected. □

Proposition 4.3. *Let $0 < \mu < \nu, \breve{p} \in (0,1)$ and $\breve{q} = \frac{\breve{p}}{\breve{p}-1}, \breve{q} > 1$. Then*

$$|A^n(\mu,\nu) - L_n^n(\mu,\nu)| \leq \frac{(\nu-\mu)}{2\breve{p}(\breve{p}+1)} + \frac{M(\nu-\mu)}{2\breve{q}}\left[\frac{2(2^{\breve{p}}-1)}{\breve{p}}\right]$$

Proof. We can derive inequality from Theorem 3.7 with $\varrho, r, \alpha, n, j \to 1$, and $\phi \to \frac{\mu+\nu}{2}$ which is applied for the $h-$convex functions that are $\breve{f} : \mathbb{R} \to \mathbb{R}, \breve{f}(\phi) = \phi^n$, and $h : \mathbb{R} \to \mathbb{R}, h(u) = (1+u)^{\breve{p}-1}$ for $\breve{p} \in (0,1)$. Further explanations can be neglected. □

Proposition 4.4. *Let $0 < \mu < \nu, \breve{p} \in (0,1)$ and $\breve{q} = \frac{\breve{p}}{\breve{p}-1}, \breve{q} > 1$. Then*

$$|A^n(\mu,\nu) - L_n^n(\mu,\nu)| \leq \frac{M(\nu-\mu)}{2}\left[\frac{\sqrt[\breve{q}]{\frac{2^{\breve{p}}-1}{\breve{p}}}(1+\sqrt[\breve{p}]{\breve{p}+1})}{\sqrt[\breve{p}]{(\breve{p}+1)(\breve{p}+2)}}\right]$$
$$\leq \frac{M(\nu-\mu)}{2\sqrt[\breve{p}]{(\breve{p}+1)}}\sqrt[\breve{q}]{\frac{2(2^{\breve{p}}-1)}{\breve{p}}}.$$

Proof. We can derive inequality from Theorem 3.10 with $\varrho, \alpha, r, n, j \to 1$, and $\phi \to \frac{\mu+\nu}{2}$ which is applied for the $h-$convex functions that are $\breve{f} : \mathbb{R} \to \mathbb{R}, \breve{f}(\phi) = \phi^n$, and $h : \mathbb{R} \to \mathbb{R}, h(u) = (1+u)^{\breve{p}-1}$ for $\breve{p} \in (0,1)$. Further explanations can be neglected. □

Proposition 4.5. *Let* $0 < \mu < \nu, \breve{p} \in (0,1)$ *and* $\breve{q} = \frac{\breve{p}}{\breve{p}-1}, \breve{q} \geq 1$. *Then*

$$|A^n(\mu,\nu) - L_n^n(\mu,\nu)| \leq \frac{M(\nu-\mu)}{2}\left[\frac{\sqrt[\breve{q}]{\frac{2(4+2^{\breve{p}+1}(\breve{p}-2)+\breve{p})}{\breve{p}(\breve{p}+1)(\breve{p}+2)}}}{\sqrt[\breve{q}]{6^{\breve{q}-1}}} + \frac{\sqrt[q]{\frac{10(2^{\breve{p}}-1)+2^{\breve{p}}(\breve{p}-1)\breve{p}-(\breve{p}+5)\breve{p}}{\breve{p}(\breve{p}+1)(\breve{p}+2)}}}{\sqrt[\breve{q}]{3^{\breve{q}-1}}}\right]$$

$$\leq \frac{M(\nu-\mu)}{2\sqrt[\breve{q}]{2^{\breve{q}-1}}}\sqrt[\breve{q}]{\frac{2^{\breve{p}}-1}{\breve{p}}}.$$

Proof. We can derive inequality from Theorem 3.13 with $\varrho, \alpha, r, n, j \to 1$, and $\phi \to \frac{\mu+\nu}{2}$ which is applied for the $h-$convex functions that are $\breve{f} : \mathbb{R} \to \mathbb{R}, \breve{f}(\phi) = \phi^n$, and $h : \mathbb{R} \to \mathbb{R}, h(u) = (1+u)^{\breve{p}-1}$ for $\breve{p} \in (0,1)$. Further explanations can be neglected. □

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5. Conclusion

We generalized Ostrowski-type fractional integral inequalities using the katugampola fractional integrals for $h-$convex. Due to the fact that the Katugampola fractional integrals are the generalization of Riemann-Liouville fractional integrals and Hadamard fractional integrals. In particular, if we take the limits as $\varrho \to 1$ and $\varrho \to 0$, then our results could be stated using the Riemann-Liouville and Hadamard fractional integrals respectively. Some particular cases have also been considered. We believe that these results will inspire further research on fractional integral inequalities and their applications

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