Soliton solutions of coupled complex modified Korteweg-de Vries system through Binary Darboux transformation

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Abstract.: In this article, we find various kind of solutions of coupled complex modified (KdV) system by using very interesting method binary Darboux transformation. Generally the solutions are classified into zero seed and non-zero seed. In zero seed solutions, we find breather solution and one soliton solution. While in non-zero seed solutions, we obtain bright-bright solitons, w-shaped solitons, bright-dark solitons, periodic and rouge waves solutions. The behavior of these solutions can easily examine from figures.

1. INTRODUCTION

In nonlinear science, integrable models play an essential role to describe the wave propagation in nonlinear media. These integrable systems have a number of applications in different phenomenons like water wave, signal transmission in optical fibers, magnetics, molecular biology and plasma physics [6, 7, 16, 25, 21, 24, 23, 1, 18]. To obtain soliton solutions of these integrable systems is very difficult task for both mathematicians and physicists. In 1882, Darboux originated the idea about Darboux transformation when he was studying linear Sturm - Liouville problem. A century later in 1970, V. B Matveev extended his idea to important different partial differential equations (PDEs).Now a days Darboux transformation is very powerful tool to get soliton solutions and analyses it. A few authors studied different integrable models like nonlinear Schrodinger equation (NLS),

Sine-Gordon equation and Korteweg-de Vries (KdV) equation and obtain soliton solutions by using Darboux transformation [27, 30, 17, 4, 5, 2].

The complex modified Korteweg-de Vries (mKdV) equation is very famous integrable model

$$A_t + A_{xxx} + 6|A|^2 A_x = 0 \tag{1. 1}$$

This integrable system has been studied by number of researchers and obtained its breather solutions, conservation laws, periodic solutions, rogue wave solutions, M and W type solutions and interaction complex solitons [26, 8, 10, 9, 20]. Also, many researchers have been studied the generalizations of mKdV and obtained multi- soliton solutions. Vector version of mKdV was proposed in [15], solved initial value problem and obtain multi-soliton solutions in [19]. Iwao and Hirota [22] give coupled complex mKdV system as

$$
\begin{aligned}
A_t + A_{xxx} + 6(|A|^2 + |B|^2)A_x &= 0, \\
B_t + B_{xxx} + 6(|A|^2 + |B|^2)B_x &= 0.
\end{aligned}
\tag{1. 2}
$$

This coupled (cmKdV) system used in orthogonally polarized transverse waves, molecular chain model and plasma waves [3, 28]. When we study coupled (cmKdV) to analyses the behavior of orthogonally polarized transverse waves, we find different geometrical orientation of oscillations. If we solve coupled (cmKdV) system and find its exact solutions than we can easily compare and control the parameters of oscillations. Similar method canbe applied to molecular chain model and plasma waves. Different methods have been applied to different integrable systems to find different types of solutions [31, 11]. Many authors obtained interesting kind of exact solutions called lump solutions [12, 29]. For constructing exact solutions of integrable nonlinear equations binary DT is very useful tool in soliton theory. In this method both spectral problem and corresponding adjoint associated with nonlinear system remain invariant by applying binary DT. Also the eigenvalues of Lax pair must be equal to the eigenvalues of its adjoint. In [13, 14], authors discuss the symmetries and reductions for the matrix mKdV and multicomponent NLS equations and find soliton solutions by using binary DT. In this article, we will use binary DT to obtain soliton solutions of system in equation (1.2). By iteration of binary DT along with quasideterminants, we will obtain multi- soliton solutions for both zero seed and nonzero seed solutions.

The sections of this article are divided as follows. In section 2, we briefly discuss the Lax pair of coupled(cmKdV) equation and standard binary DT. In section 3, many types of solutions are obtained for both zero seed and nonzero seed. The categories of these solutions are breather solutions, periodic solutions, rogue wave solutions, bright-dark solutions, bright-w-shaped solitons and bright-bright solitons.

2. Lax pair and Binary DT

2.1. **Symmetries of Lax pair.** The Lax pair of system in equation (1.2) is

$$
\begin{aligned}
\Psi_x &= -(P - \partial_x)\Psi, \\
\Psi_t &= -(Q - \partial_t)\Psi.
\end{aligned}
\tag{2. 3}
$$

Here

$$P = \partial_x + \lambda J + R, \qquad Q = \partial_t + 4\lambda^3 J + 4\lambda^2 R - 2\lambda M + W,$$

With

$$J = i\begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \quad R = \begin{pmatrix} O & -H \\ H^\dagger & O \end{pmatrix}, \quad H = \begin{pmatrix} A^* & -B \\ B^* & A \end{pmatrix},$$

$$M = i\begin{pmatrix} HH^\dagger & H_x \\ H_x^\dagger & -HH^\dagger \end{pmatrix}, \quad W = \begin{pmatrix} H_x H^\dagger - HH_x^\dagger & H_{xx} + 2HH^\dagger H \\ -H_{xx}^\dagger - 2H^\dagger HH^\dagger & H_x^\dagger H - H^\dagger H_x \end{pmatrix}.$$

Where $\dagger$ represent the Hermitian conjugate and * denoted the complex conjugate. Also I denoted 2×2 identity matrix and O represent 2×2 zero matrix. The integrability condition $\Psi_{xt} = \Psi_{tx}$ give the system in equation (1.2).

Proposition 1. *If $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)^T$ is eigenfunction having eigenvalue λ, then $\zeta = (\eta_2^*, -\eta_1^*, \eta_4^*, -\eta_3^*)^T$ is also eigenfunction of Lax pair in equation (2.3) having eigenvalue $-\lambda^*$.*

So, we can write 4×2 matrix eigenfunction Λ with 2×2 eigenvalue matrix θ as

$$\Lambda = \begin{pmatrix} \eta_1 & \eta_2^* \\ \eta_2 & -\eta_1^* \\ \eta_3 & \eta_4^* \\ \eta_4 & -\eta_3^* \end{pmatrix}, \qquad \theta = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^* \end{pmatrix} \tag{2.4}$$

Which are satisfying

$$\Lambda_x + J\Lambda\theta + R\Lambda = 0, \qquad \Lambda_t + 4J\Lambda\theta^3 - 2M\Lambda\theta + W\Lambda = 0 \tag{2.5}$$

2.2. **Binary Darboux Transformation.** The linear differential operators are given as

$$P = \partial_x + \sum_{i=0}^{N} A_i \partial_y^i, \qquad Q = \partial_t + \sum_{i=0}^{N} B_i \partial_y^i \tag{2.6}$$

Here A_i and B_i are $m \times m$ complex matrices.

The standard Darboux transformation can be presented as.

Theorem 1. *The linear system is*

$$P(\eta) = Q(\eta) = 0 \tag{2.7}$$

If Λ be non-singular $m \times m$ matrix solution of equation (2.7).Then Darboux transformation $\widetilde{\eta} = H_\Lambda(\eta) = \Lambda\partial_y\Lambda^{-1}\eta$ leaves the system in equation (2.7) invariant.

$$\widetilde{P}(\widetilde{\eta}) = \widetilde{Q}(\widetilde{\eta}) = 0 \tag{2.8}$$

The operators P and Q have same values as $\widetilde{P} = H_\Lambda P H_\Lambda^{-1}$ and $\widetilde{Q} = H_\Lambda Q H_\Lambda^{-1}$ respectively. So,

$$\widetilde{P} = \partial_x + \sum_{i=0}^{N} \widetilde{A}_i \partial_y^i, \qquad \widetilde{Q} = \partial_t + \sum_{i=0}^{N} \widetilde{B}_i \partial_y^i \tag{2.9}$$

By composing a DT with the inverse of another relative a binary DT is constructed as.

Theorem 2. *Let Λ and σ be $m \times k$ matrix solution of system in equation (2.7) and it adjoint* $P^\dagger(\zeta) = Q^\dagger(\zeta) = 0$ *respectively. Then, a binary DT*

$$B_{\Lambda,\sigma} = I - \Lambda \Upsilon(\Lambda,\sigma)^{-1}\Upsilon(\cdot,\sigma) \tag{2.10}$$

$$B_{\Lambda,\sigma}^{-\dagger} = I - \sigma \Upsilon(\Lambda,\sigma)^{-\dagger}\Upsilon(\Lambda,\cdot)^{\dagger} \tag{2.11}$$

Here $\Upsilon(\Lambda,\sigma)_y = \sigma^\dagger\Lambda$ *preserve the system in equation (2.7) and it adjoint* $P^\dagger(\zeta) = Q^\dagger(\zeta) = 0$ *respectively. Nth-iterated binary DT is constructed as.*

Theorem 3. *Let $\Lambda_1, \Lambda_2, ..., \Lambda_N$ be N linearly independent solutions of system in equation (2.7) and $\sigma_1, \sigma_2, ..., \sigma_N$ be N linearly independent solution of it adjoint* $P^\dagger(\zeta) = Q^\dagger(\zeta) = 0$. *Then N-fold binary DT can be denoted in the term of quasideterminants as*

$$\eta[N] = \left| \begin{array}{cc} \Upsilon(\Theta,\Sigma) & \Upsilon(\eta,\Sigma) \\ \Theta & \boxed{\eta} \end{array} \right|,$$

$$\zeta[N] = \left| \begin{array}{cc} \Upsilon(\Theta,\Sigma)^\dagger & \Upsilon(\Theta,\zeta)^\dagger \\ \Sigma & \boxed{\zeta} \end{array} \right|. \tag{2.12}$$

Where $\Theta = (\Lambda_1, \Lambda_2, ..., \Lambda_N)$, $\Sigma = (\sigma_1, \sigma_2, ..., \sigma_N)$, $\Upsilon(\Theta,\Sigma) = \Upsilon((\Lambda_i,\sigma_i))_{ij=1,2,...N}$ *is $N \times N$ matrix,* $\Upsilon(\eta,\Sigma) = (\Upsilon(\eta,\sigma_j))_{j=1,2,...,N}$ *is $N \times 1$ vector and* $\Upsilon(\Theta,\zeta) = (\Upsilon(\zeta,\Lambda_i))_{i=1,2,...N}$ *is a $1 \times N$ vector.*

By using separation of variables technique we reduce binary DT from (2+1) to (1+1) dimensions.

$$\eta = \eta^d(x,t)e^{\lambda y}, \quad \Lambda = \Lambda^d(x,t)e^{\theta y},$$

$$\zeta = \zeta^d(x,t)e^{\mu y}, \quad \sigma = \sigma^d(x,t)e^{\Pi y} \tag{2.13}$$

Where θ, Π are $N \times N$ constant matrices and λ, μ are constant scalars. So, matrix operators P and Q becomes

$$P^d = \partial_x + \sum_{i=0}^{N} A_i\lambda^i, \quad Q^d = \partial_t + \sum_{i=0}^{N} B_i\lambda^i \tag{2.14}$$

From above y dependence potential Υ write explicitly as

$$\Upsilon(\Lambda,\sigma) = e^{\Pi^\dagger y}\Upsilon^d(\Lambda^d,\sigma^d)e^{\theta y}, \quad \Upsilon(\eta,\sigma) = e^{(\Pi^\dagger + \lambda I)y}\Upsilon^d(\eta^d,\sigma^d),$$

Π and θ satisfy

$$\Pi^\dagger\Upsilon^d(\Lambda^d,\sigma^d) + \Upsilon^d(\Lambda^d,\sigma^d) = \sigma^{d\dagger}\Lambda^d, \quad (\Pi^\dagger + \lambda I)\Upsilon^d(\eta^d,\sigma^d) = \sigma^{d\dagger}\eta^d \tag{2.15}$$

We discuss reduced case and ignore superscript d. If the constant matrices to be diagonal $\Pi = diag(\mu_1, \mu_2, ..., \mu_N)$ and $\Theta = diag(\lambda_1, \lambda_2, ..., \lambda_N)$ then we obtain as

$$\Upsilon(\Lambda,\sigma) = \frac{(\sigma^\dagger\Lambda)_{ij}}{\lambda_j + \mu_i^*}, \quad \Upsilon(\eta,\sigma) = \frac{(\sigma^\dagger\eta)_{ij}}{\lambda_j + \mu_i^*}, \quad (i,j = 1,2,....,N) \tag{2.16}$$

As J and R are skew-Hermitian, then by using property $P + P^\dagger = 0$, $Q + Q^\dagger = 0$ keep constraints among potentials in R, we choose $\sigma = \Lambda$ and $\Pi = -\theta$ and also P is invariant under binary DT

$$P \rightarrow \widetilde{P} = B_{\Lambda,\sigma}PB_{\Lambda,\sigma}^{-1} = \partial_x + \lambda J + \widetilde{R} \tag{2.17}$$

Here

$$\widetilde{R} = R + \left[J, \Lambda \Upsilon(\Lambda, \Lambda)^{-1} \Lambda^\dagger \right] \tag{2.18}$$

For simplicity, we introduce 4×4 matrix

$$S = \frac{1}{2i} \begin{pmatrix} O & H \\ H^\dagger & O \end{pmatrix} \tag{2.19}$$

and $R = [S, J]$ then

$$\widetilde{S} = S + \begin{vmatrix} \Upsilon(\Lambda, \Lambda) & \Lambda^\dagger \\ \Lambda & \boxed{O} \end{vmatrix} = S - \Lambda \Upsilon(\Lambda, \Lambda)^{-1} \Lambda^\dagger \tag{2.20}$$

Here $\Upsilon(\Lambda, \Lambda)$ satisfy property

$$\Upsilon(\Lambda, \Lambda)\theta - \Lambda^\dagger \Upsilon(\Lambda, \Lambda) = \Lambda^\dagger \Lambda \tag{2.21}$$

Theorem 4. *Let $\Lambda_1, \Lambda_2, ..., \Lambda_N$ are N different linearly independent solutions of system $P(\eta) = Q(\eta) = 0$ with $\lambda_1, \lambda_2, ..., \lambda_N$. Then N-fold iterative potential transformation becomes*

$$S[N] = S + \begin{vmatrix} \Upsilon(\Theta, \Theta) & \Theta^\dagger \\ \Theta & \boxed{O} \end{vmatrix} \tag{2.22}$$

Where

$$\Upsilon(\Theta, \Theta) = \begin{pmatrix} \Upsilon(\Lambda_1, \Lambda_1) & \Upsilon(\Lambda_2, \Lambda_1) & \cdot & \cdot & \cdot & \Upsilon(\Lambda_N, \Lambda_1) \\ \Upsilon(\Lambda_1, \Lambda_2) & \Upsilon(\Lambda_2, \Lambda_2) & \cdot & \cdot & \cdot & \Upsilon(\Lambda_N, \Lambda_2) \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \Upsilon(\Lambda_1, \Lambda_N) & \Upsilon(\Lambda_2, \Lambda_N) & \cdot & \cdot & \cdot & \Upsilon(\Lambda_N, \Lambda_N) \end{pmatrix}$$

With

$$\Theta = (\Lambda_1, \Lambda_2, ..., \Lambda_N) \quad, \Lambda_k = \begin{pmatrix} \eta_{4k-3} & \eta^*_{4k-2} \\ \eta_{4k-2} & -\eta^*_{4k-3} \\ \eta_{4k-1} & \eta^*_{4k} \\ \eta_{4k} & -\eta^*_{4k-1} \end{pmatrix} \quad, \quad k = (1, 2, ..., N)$$

Here $\Upsilon(\Lambda_i, \Lambda_j)$ satisfy

$$\Upsilon(\Lambda_i, \Lambda_j)\theta_i - \theta_j^\dagger \Upsilon(\Lambda_i, \Lambda_j) = \Lambda_j^\dagger \Lambda_i, \quad \theta_i = diag(\lambda_i, -\lambda_i^*), \qquad (i, j = 1, 2, ..., N) \tag{2.23}$$

Potential Υ can be written explicitly from above

$$\Upsilon(\Lambda_i, \Lambda_j) = \begin{pmatrix} F_{ij} & H^*_{ij} \\ H_{ij} & -F_{ij} \end{pmatrix} \tag{2.24}$$

Where

$$\begin{aligned} F_{ij} &= \frac{1}{\lambda_i - \lambda_j^*}(\eta_{4i}\eta^*_{4j} + \eta_{4i-1}\eta^*_{4j-1} + \eta_{4i-2}\eta^*_{4j-2} + \eta_{4i-3}\eta^*_{4j-3}), \\ G_{ij} &= \frac{1}{\lambda_i - \lambda_j}(\eta_{4i-3}\eta_{4j-2} - \eta_{4i-2}\eta_{4j-3} + \eta_{4i-1}\eta_{4j} + \eta_{4i}\eta_{4j-1}). \quad (2.25) \end{aligned}$$

Inserting eq. (2.24) into eq. (2.22) we get

$$
A\left[N\right] = A + 2i\begin{vmatrix} \Upsilon(\Theta,\Theta) & \Psi_1^\dagger \\ \Psi_3 & \boxed{O} \end{vmatrix} = A + 2i\begin{vmatrix} \Upsilon(\Theta,\Theta) & \Psi_4^\dagger \\ \Psi_2 & \boxed{O} \end{vmatrix},
$$

$$
B\left[N\right] = B + 2i\begin{vmatrix} \Upsilon(\Theta,\Theta) & \Psi_2^\dagger \\ \Psi_3 & \boxed{O} \end{vmatrix} = B - 2i\begin{vmatrix} \Upsilon(\Theta,\Theta) & \Psi_4^\dagger \\ \Psi_1 & \boxed{O} \end{vmatrix},
$$

$$
A\left[N\right]^* = A^* + 2i\begin{vmatrix} \Upsilon(\Theta,\Theta) & \Psi_3^\dagger \\ \Psi_1 & \boxed{O} \end{vmatrix} = A^* + 2i\begin{vmatrix} \Upsilon(\Theta,\Theta) & \Psi_2^\dagger \\ \Psi_4 & \boxed{O} \end{vmatrix},
$$

$$
B\left[N\right]^* = B^* + 2i\begin{vmatrix} \Upsilon(\Theta,\Theta) & \Psi_3^\dagger \\ \Psi_2 & \boxed{O} \end{vmatrix} = B^* - 2i\begin{vmatrix} \Upsilon(\Theta,\Theta) & \Psi_1^\dagger \\ \Psi_4 & \boxed{O} \end{vmatrix}. \quad (2.\,26)
$$

Here $\Psi_j = (j = 1,2,3,4)$ is jth row of matrix Θ. We can easily check these relation by applying quasideterminants properties.

3. ZERO SEED AND NON-ZERO SEED SOLUTIONS OF COUPLED (CMKDV) SYSTEM

We construct different exact solutions in this section for both zero seed and non-zero seed of coupled (cmKdV) system.

3.1. **Zero seed solutions.** By using quasideterminants properties to the transformations in equation (2.26) one can easily obtain once-iterated transformations as

$$
[A] = A - 2i\frac{(\lambda_1 - \lambda_1^*)(\eta_1^*\eta_3 + \eta_2\eta_4^*)}{|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2 + |\eta_4|^2},
$$

$$
[B] = B - 2i\frac{(\lambda_1 - \lambda_1^*)(\eta_2^*\eta_3 + \eta_1\eta_4^*)}{|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2 + |\eta_4|^2}. \quad (3.\,27)
$$

The system in equation (2.7) become when $A = B = 0$ of equation (1.2) for zero seed solutions.

$$
\begin{aligned}
\eta_x + \lambda J\eta &= 0, \\
\eta_t + 4\lambda^3 J\eta &= 0.
\end{aligned} \quad (3.\,28)
$$

having solution

$$
\eta = (\alpha e^{-i\rho}, \beta e^{-i\rho}, \gamma e^{i\rho}, \delta e^{i\rho})^T, \quad \rho = \lambda(x + 4\lambda^2 t) \quad (3.\,29)
$$

3.1.1. *Breather solution.* When $N = 1$, by substituting solution $\eta^{[1]} = (\eta_1, \eta_2, \eta_3, \eta_4)^T = (\alpha_1 e^{-i\rho_1}, \beta_1 e^{-i\rho_1}, \gamma_1 e^{i\rho_1}, \delta_1 e^{i\rho_1})^T$ with $\rho_1 = \lambda_1(x + 4\lambda^2 t)$ where $\lambda_1 = a_1 + ib_1, (b_1 \neq 0)$

in equation (3.27) one can get breather solution of equation (1.2) as

$$
\begin{aligned}
A[1] &= \frac{4b_1[(\alpha_1^*\gamma_1 + \beta_1\delta_1^*)\cos(-2a_1[x + 4(a_1^2 - 3b_1^2)t])}{(|\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 + |\delta_1|^2)\cosh(-2b_1[x + 4(3a_1^2 - b_1^2)t])} \\
&\quad \frac{+i(\beta_1\delta_1^* - \alpha_1^*\gamma_1)\sin(-2a_1[x + 4(a_1^2 - 3b_1^2)t])]}{+(|\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 + |\delta_1|^2)\sinh(-2b_1[x + 4(3a_1^2 - b_1^2)t])}, \\
B[1] &= \frac{4b_1[(\alpha_1\delta_1^* - \beta_1^*\gamma_1)\cos(-2a_1[x + 4(a_1^2 - 3b_1^2)t])}{(|\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 + |\delta_1|^2)\cosh(-2b_1[x + 4(3a_1^2 - b_1^2)t])} \\
&\quad \frac{+i(\alpha_1\delta_1^* + \beta_1^*\gamma_1)\sin(-2a_1[x + 4(a_1^2 - 3b_1^2)t])]}{+(|\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 + |\delta_1|^2)\sinh(-2b_1[x + 4(3a_1^2 - b_1^2)t])}.(3.\,30)
\end{aligned}
$$

For simplicity we use notations as

$$
\begin{aligned}
\upsilon &= -2a_1[x + 4(a_1^2 - 3b_1^2)t], \quad \varsigma = -2b_1[x + 4(3a_1^2 - b_1^2)t], \\
l_1 &= |\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 + |\delta_1|^2, \quad l_2 = (|\gamma_1|^2 + |\delta_1|^2 - |\alpha_1|^2 - |\beta_1|^2)(3.\,31)
\end{aligned}
$$

Now the solution in equation (3.30) changed to

$$
\begin{aligned}
A[1] &= \frac{4b_1[(\alpha_1^*\gamma_1 + \beta_1\delta_1^*)\cos(\upsilon) + i(\beta_1\delta_1^* - \alpha_1^*\gamma_1)\sin(\upsilon)]}{l_1\cosh(\varsigma) + l_2\sinh(\varsigma)}, \\
B[1] &= \frac{4b_1[(\alpha_1\delta_1^* - \beta_1^*\gamma_1)\cos(\upsilon) + i(\alpha_1\delta_1^* + \beta_1^*\gamma_1)\sin(\upsilon)]}{l_1\cosh(\varsigma) + l_2\sinh(\varsigma)}. \quad (3.\,32)
\end{aligned}
$$
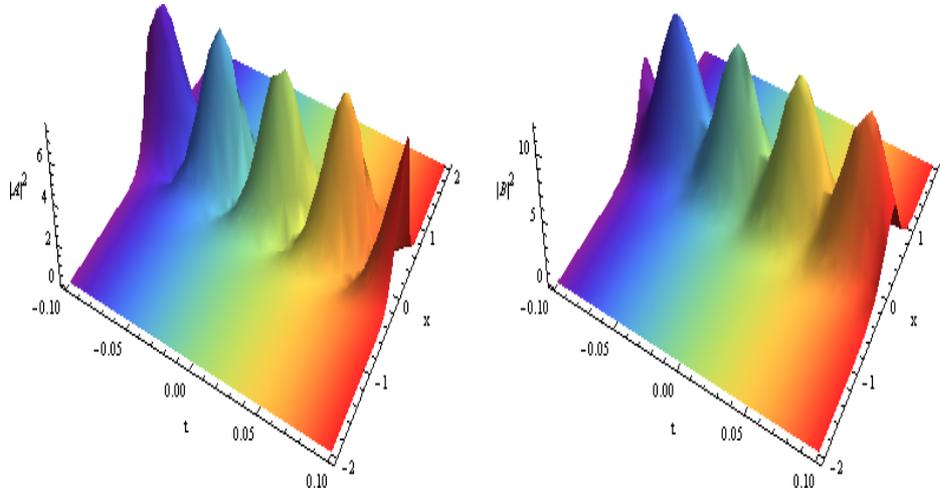
Figure:1 Breather solutions having parameters:$\alpha_1 = \frac{1}{2}, \beta_1 = \delta_1 = 1, \gamma_1 = 2, \lambda_1 = \sqrt{3}i + 1$

3.1.2. *Bright one-soliton solutions.* The breather solutions in equation (3.32) are defined by parameters $a_1, b_1, \alpha_1, \beta_1, \gamma_1$ and δ_1. If we take zero any one of these then we obtain

bright one-soliton solution. we take $\alpha_1 = 0$ to get the following solutions as

$$
\begin{aligned}
A[1] &= \frac{(4b_1\beta_1\delta_1^*)e^{iv}}{|\beta_1|^2 + |\gamma_1|^2 + |\delta_1|^2 \cosh(\varsigma)\,|\gamma_1|^2 + |\delta_1|^2 - |\beta_1|^2 \sinh(\varsigma)}, \\
B[1] &= \frac{(4b_1\beta_1^*\gamma_1)e^{-iv}}{|\beta_1|^2 + |\gamma_1|^2 + |\delta_1|^2 \cosh(\varsigma)\,|\gamma_1|^2 + |\delta_1|^2 - |\beta_1|^2 \sinh(\varsigma)}.
\end{aligned} \quad (3.\,33)
$$

If

$$
l_3 = |\beta_1|^2 + |\gamma_1|^2 + |\delta_1|^2, \quad l_4 = |\gamma_1|^2 + |\delta_1|^2 - |\beta_1|^2 \quad (3.\,34)
$$

Then the solutions becomes

$$
\begin{aligned}
A[1] &= \frac{(4b_1\beta_1\delta_1^*)e^{iv}}{l_3 \cosh(\varsigma)\, l_4 \sinh(\varsigma)}, \\
B[1] &= \frac{(4b_1\beta_1^*\gamma_1)e^{-iv}}{l_3 \cosh(\varsigma)\, l_4 \sinh(\varsigma)}.
\end{aligned} \quad (3.\,35)
$$
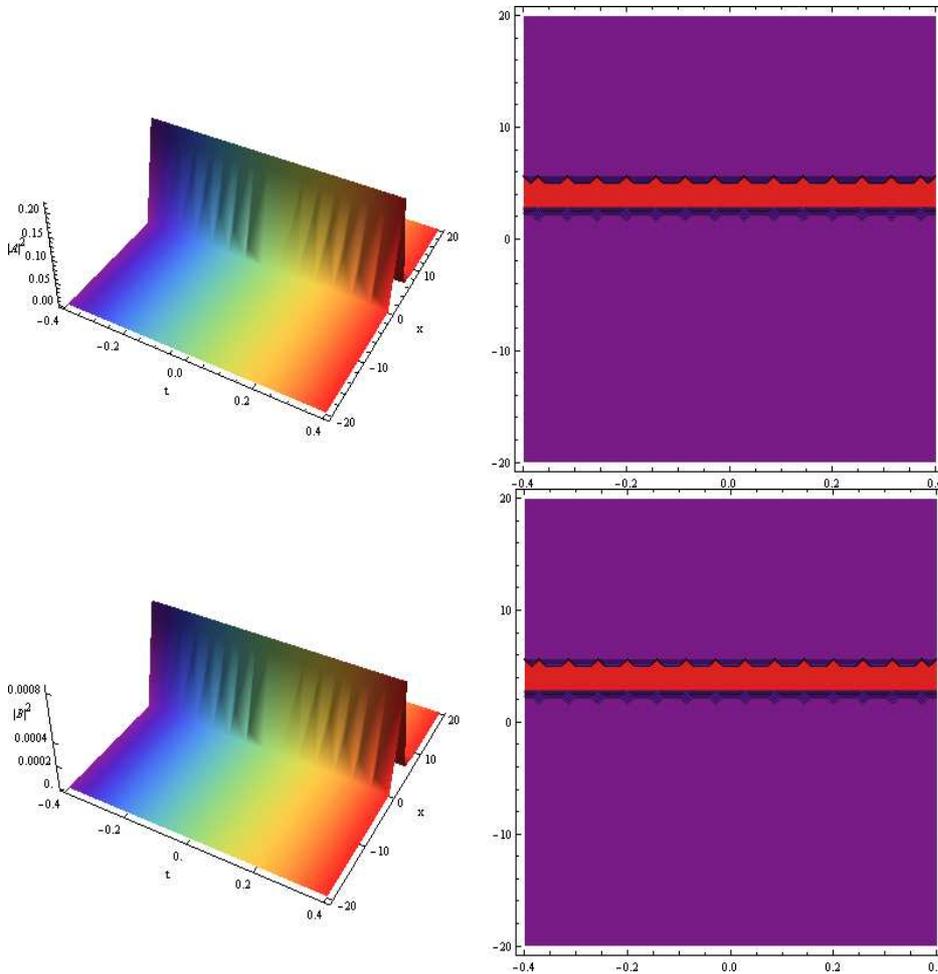
Figure:2 Bright one-soliton with parameters:$\alpha_1 = 2, \beta_1 = 1, \gamma_1 = \frac{1}{2}, \delta_1 = 0$ and $\lambda_1 = i + \frac{1}{8}$

By using same algorithm we can easily obtain two-soliton solution when $N = 2$.

3.2. **Non-zero seed solutions.**

3.2.1. *Case 1.* Let $g_j (j = 1, 2)$ is a complex constants for non-zero seed solution of A and B. Inserting A and B into equation (2.7) we get the solution as $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)^T$, where

$$
\begin{align}
\eta_1 &= \beta e^\vartheta + \gamma e^{-\vartheta}, & (3.\,36) \\
\eta_2 &= \delta e^\vartheta + \alpha e^{-\vartheta}, & (3.\,37) \\
\eta_3 &= e^\vartheta \varrho^+ (g_2 \delta + g_1 \beta) + e^{-\vartheta} \varrho^- (g_2 \alpha + g_1 \gamma), & (3.\,38) \\
\eta_4 &= e^\vartheta \varrho^+ (g_1^* \delta + g_2^* \beta) + e^{-\vartheta} \varrho^- (g_1^* \alpha + g_2^* \gamma). & (3.\,39)
\end{align}
$$

Where

$$
\begin{align}
\vartheta &= i \sqrt{|g_1|^2 + |g_2|^2 + \lambda^2} \left[x + 2(2\lambda^2 - \left(|g_1|^2 + |g_2|^2\right)) t \right], \\
\varrho^\pm &= \frac{i\lambda \pm i \sqrt{\lambda^2 + |g_1|^2 + |g_2|^2}}{|g_1|^2 + |g_2|^2}. & (3.\,40)
\end{align}
$$

We can get the solutions of equation (1.2) by substituting equation (3.36-3.39) into equation (3.27). Here we choose $\lambda_1 = ih$ ($h \neq 0$). This leads to two different cases for parameters values.

(1). If $h > \sqrt{|g_1|^2 + |g_2|^2}$, then the terms in equation (3.40) changed to

$$
\begin{align}
\vartheta &= -\sqrt{h^2 - \left(|g_1|^2 + |g_2|^2\right)} \left[x - 2(2h^2 + \left(|g_1|^2 + |g_2|^2\right)) t \right], \\
\varrho^\pm &= \frac{-h \mp \sqrt{h^2 - \left(|g_1|^2 + |g_2|^2\right)}}{|g_1|^2 + |g_2|^2}. & (3.\,41)
\end{align}
$$

Upon calculation, we obtain

$$
\begin{align}
A[1] &= g_1 \left(1 - \frac{4hE_1}{g_1 k} \right), & (3.\,42) \\
B[1] &= g_2 \left(1 - \frac{4hE_2}{g_2 k} \right). & (3.\,43)
\end{align}
$$

Where

$$
\begin{aligned}
E_1 &= -e^{2\vartheta}\varrho^+ g_1 \left(|\beta|^2 + |\delta|^2\right) - e^{-2\vartheta}\varrho^- g_1 \left(|\alpha|^2 + |\gamma|^2\right) \\
&\quad -\varrho^+ \left[g_1 \left(\beta\gamma^* + \delta\alpha^*\right) + g_2 \left(\delta\gamma^* - \beta^*\alpha\right)\right] \\
&\quad -\varrho^- \left[g_1 \left(\gamma\beta^* + \alpha^*\delta\right) + g_2 \left(\alpha\beta^* - \gamma^*\delta\right)\right], \qquad (3.44)\\
E_2 &= -e^{2\vartheta}\varrho^+ g_2 \left(|\beta|^2 + |\delta|^2\right) - e^{-2\vartheta}\varrho^- g_2 \left(|\alpha|^2 + |\gamma|^2\right) \\
&\quad -\varrho^+ \left[g_1 \left(\beta\alpha^* + \delta^*\gamma\right) + g_2 \left(\delta\alpha^* - \beta^*\gamma\right)\right] \\
&\quad -\varrho^- \left[g_1 \left(\gamma\delta^* + \alpha^*\beta\right) + g_2 \left(\alpha\delta^* - \gamma^*\beta\right)\right], \qquad (3.45)\\
k &= e^{2\vartheta}\left(|\beta|^2 + |\delta|^2\right)\left[1 + \varrho^+\varrho^+ \left(|g_1|^2 + |g_2|^2\right)\right] \\
&\quad + e^{-2\vartheta}\left(|\gamma|^2 + |\alpha|^2\right)\left[1 + \varrho^-\varrho^- \left(|g_1|^2 + |g_2|^2\right)\right] \\
&\quad + \left(\beta\gamma^* + \beta^*\gamma + \delta\alpha^* + \delta^*\alpha\right)\left[1 + \varrho^+\varrho^- \left(|g_1|^2 + |g_2|^2\right)\right]. \qquad (3.46)
\end{aligned}
$$

By choosing suitable parameters we obtain three kinds of solution of equation (3.42-3.43) as

Bright-bright soliton solution: If the parameters are $\alpha = \beta = \gamma = \delta = g_1 = 1, g_2 = -i$ and $h = 2$ then both equation (3.42-3.43) give bright-bright solitons.
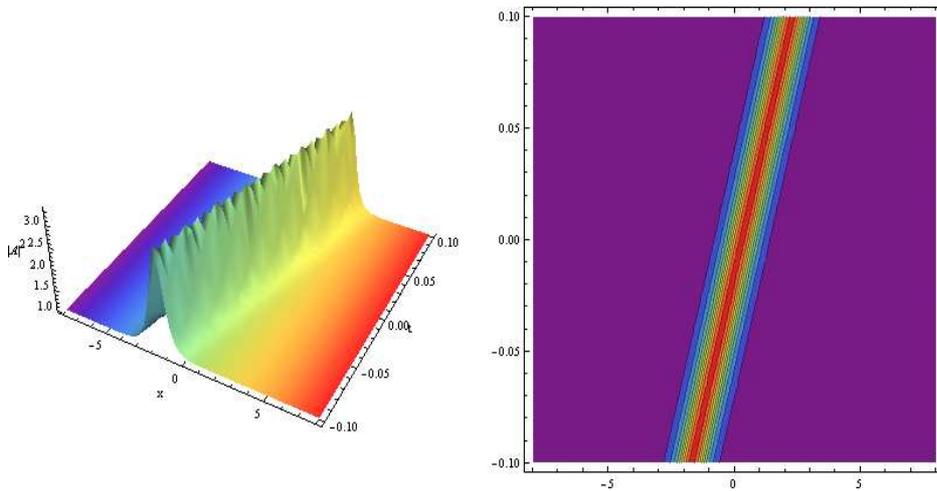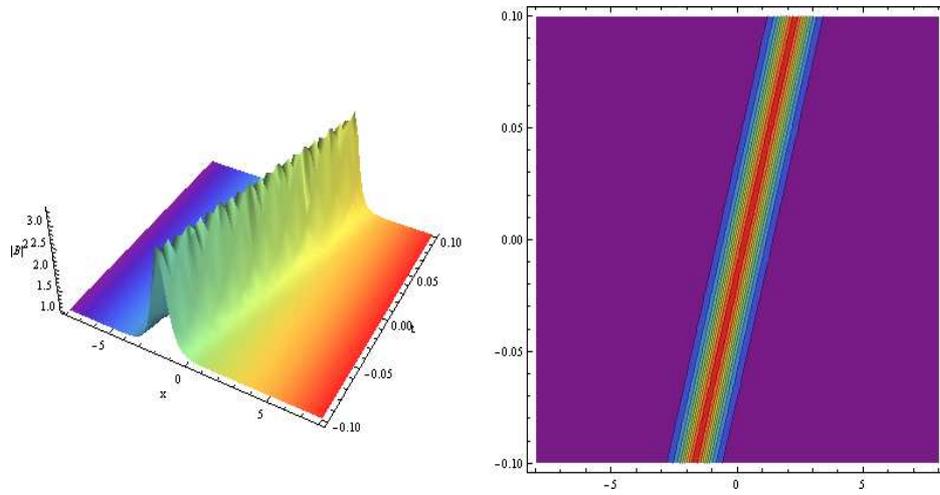
Figure:3 Bright-bright solitons having parameters $\alpha = \beta = \gamma = \delta = g_1 = 1, g_2 = -i, h = 2$

Bright-dark soliton solution: If the parameters are $\alpha = \gamma = \delta = g_1 = 1, \beta = 0, g_2 = -1$ and $h = 2$ then solution equation (3.42) is dark and equation (3.43) is bright.
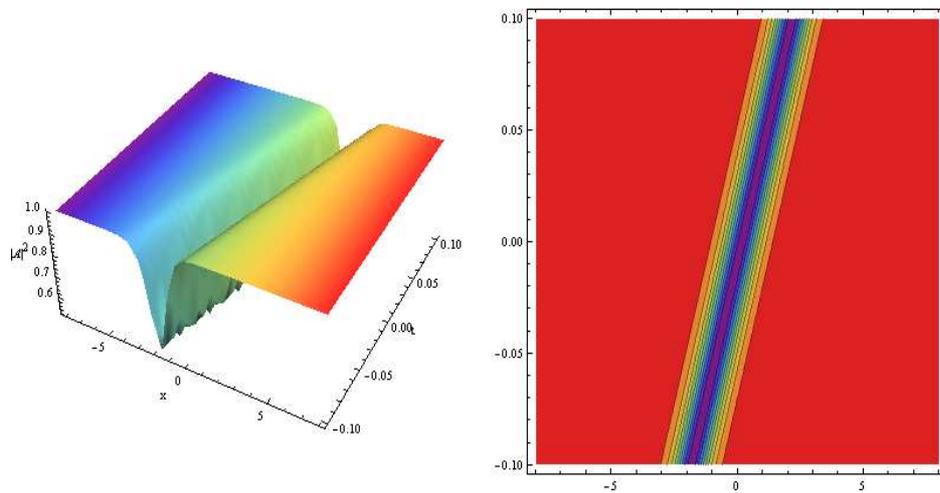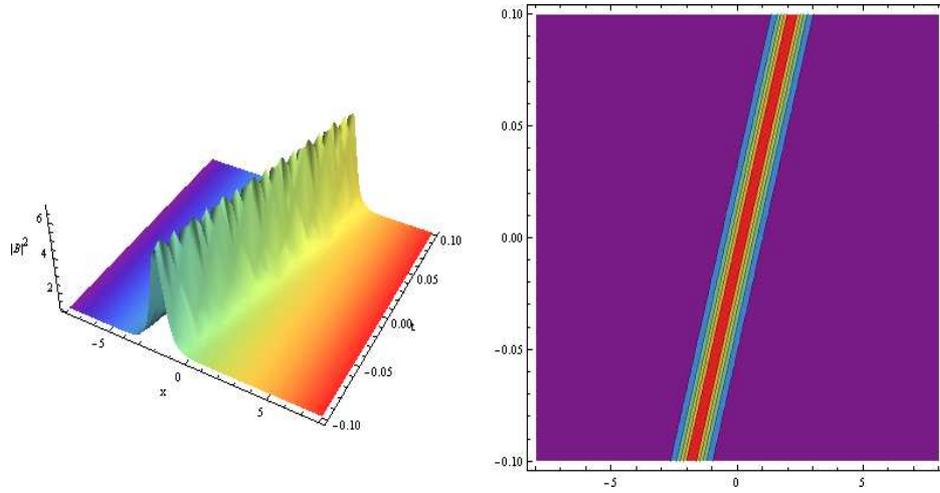
Figure:4 Bright-dark solitons having parameters $\alpha = \gamma = \delta = g_1 = 1, \beta = 0, g_2 = -1, h = 2$

W-shaped soliton solution: If the parameters are $\alpha = \beta = \delta = g_1 = 1, g_2 = 2, h = \sqrt{15}$ then the solution in equation (3.42) is w-shaped and (3.43) is bright solitons.
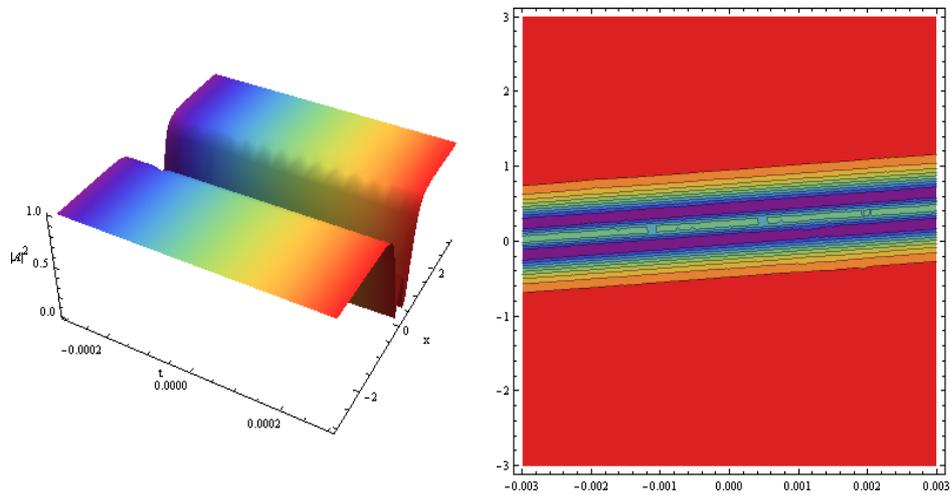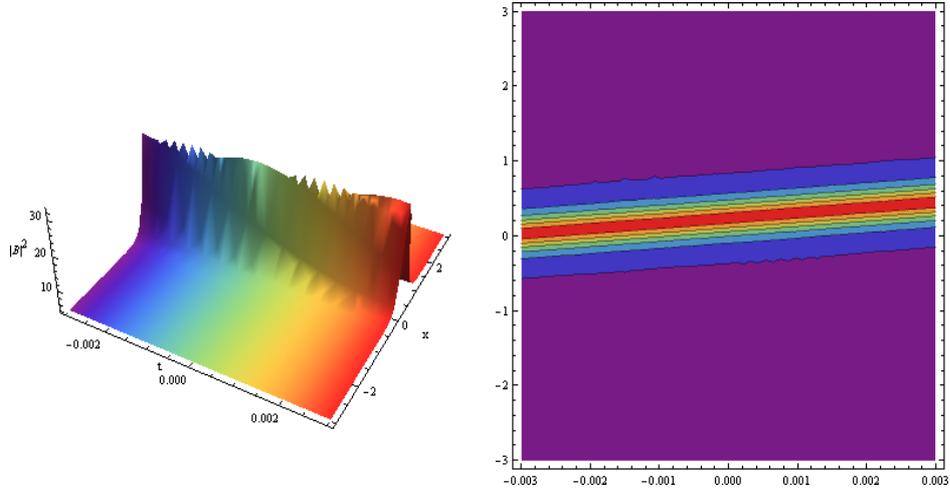
Figure:5 having parameters $\alpha = \beta = \delta = g_1 = 1, g_2 = 2, h = \sqrt{15}$

(2) If $h < \sqrt{|g_1|^2 + |g_2|^2}$, then the terms in equation (3.40) changed to

$$\vartheta = i\sqrt{\left(|g_1|^2 + |g_2|^2\right) - h^2}\left[x - 2(2h^2 + \left(|g_1|^2 + |g_2|^2\right)t\right],$$

$$\varrho^{\pm} = \frac{-h \pm i\sqrt{\left(|g_1|^2 + |g_2|^2\right) - h^2}}{|g_1|^2 + |g_2|^2}. \tag{3.47}$$

Upon calculation, we obtain

$$A[1] = g_1\left(1 - \frac{4hC_1}{g_1\Gamma}\right), \tag{3.48}$$

$$B[1] = g_2\left(1 - \frac{4hC_2}{g_2\Gamma}\right). \tag{3.49}$$

Here

$$
\begin{aligned}
C_1 &= -e^{2\vartheta} \varrho^+ g_1 \left(\gamma^* \beta + \alpha^* \delta\right) - e^{-2\vartheta} \varrho^- g_1 \left(\gamma \beta^* + \alpha \delta^*\right) \\
&\quad - \varrho^+ \left[g_1 \left(|\beta|^2 + |\alpha|^2\right) + g_2 \left(\beta^* \delta - \gamma^* \alpha\right) \right] \\
&\quad - \varrho^- \left[g_1 \left(|\gamma|^2 + |\delta|^2\right) + g_2 \left(\alpha \gamma^* - \beta^* \delta\right) \right],
\end{aligned}
\tag{3.50}
$$

$$
\begin{aligned}
C_2 &= -e^{2\vartheta} \varrho^+ g_1 \left(\gamma^* \beta + \alpha^* \delta\right) - e^{-2\vartheta} \varrho^- g_2 \left(\gamma \beta^* + \alpha \delta^*\right) \\
&\quad - \varrho^+ \left[g_2 \left(|\gamma|^2 + |\delta|^2\right) + g_1 \left(\beta \delta^* - \gamma \alpha^*\right) \right] \\
&\quad - \varrho^- \left[g_2 \left(|\beta|^2 + |\alpha|^2\right) + g_1 \left(\alpha^* \gamma - \beta \delta^*\right) \right],
\end{aligned}
\tag{3.51}
$$

$$
\begin{aligned}
\Gamma &= e^{2\vartheta} \left(\delta \alpha^* + \beta \gamma^*\right) \left[1 + \varrho^+ \varrho^+ \left(|g_1|^2 + |g_2|^2\right) \right] \\
&\quad + e^{-2\vartheta} \left(\delta^* \alpha + \beta^* \gamma\right) \left[1 + \varrho^- \varrho^- \left(|g_1|^2 + |g_2|^2\right) \right] \\
&\quad + \left(|\alpha|^2 + |\gamma|^2 + |\delta|^2 + |\beta|^2\right) \left[1 + \varrho^+ \varrho^- \left(|g_1|^2 + |g_2|^2\right) \right].
\end{aligned}
\tag{3.52}
$$

With parameters $\alpha = \beta = \delta = g_1 = g_2 = 1, \gamma = 0$ and $h = 1$, we obtain periodic solutions of equation (3.48-3.49) shown in figure 6.
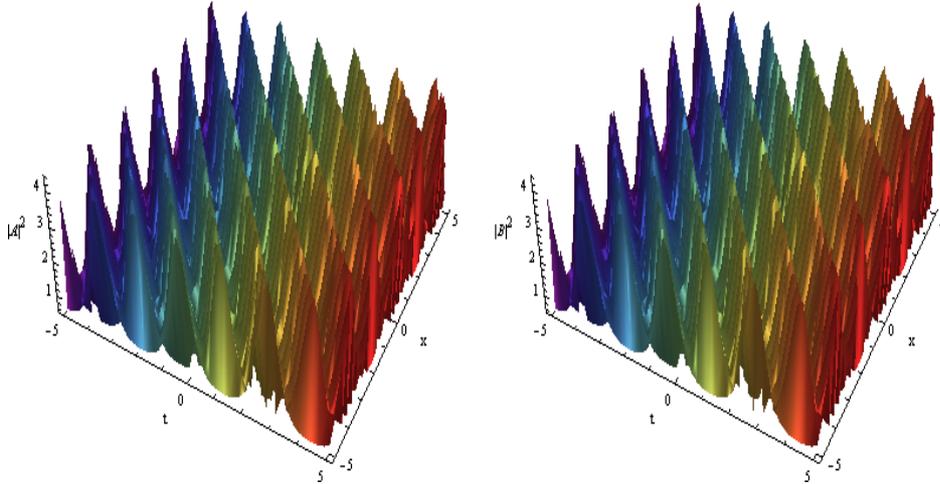
Figure:6 parameters $\alpha = \beta = \delta = g_1 = g_2 = 1, \gamma = 0$ and $h = 1$

3.3. **Case 2.** In this case following generalized DT we get rogue waves solutions of coupled (cmKdV) system. We start with plane wave seed solution

$$
\begin{aligned}
A &= g_1 e^{i\epsilon_1}, \quad \epsilon_1 = f_1 x + \left[f_1^3 - 6\left(g_1^2 + g_2^2\right) f_1\right] t, \\
B &= g_2 e^{i\epsilon_2}, \quad \epsilon_2 = f_2 x + \left[f_2^3 - 6\left(g_1^2 + g_2^2\right) f_2\right] t.
\end{aligned}
\tag{3.53}
$$

We get special solution as after inserting

$$\Psi(\lambda) = \begin{pmatrix} \alpha e^{i\chi_1} + \beta e^{i\chi_2} \\ \left(\alpha\delta e^{i\chi_1} - \beta\delta e^{i\chi_2}\right) e^{i(\epsilon_1 - \epsilon_2)} \\ \left(\alpha\frac{-g_1 - g_2\delta}{if_1 + i\tau_1 - i\lambda}e^{i\chi_1} + \beta\frac{-g_1 - g_2\delta}{if_1 + i\tau_2 - i\lambda}e^{i\chi_2}\right) e^{i\epsilon_1} \\ \left(\alpha\frac{-g_2 - g_1\delta}{if_2 + i\tau_1 - i\lambda}e^{i\chi_1} + \beta\frac{-g_2 - g_1\delta}{if_2 + i\tau_2 - i\lambda}e^{i\chi_2}\right) e^{-i\epsilon_2} \end{pmatrix} \tag{3.54}$$

Where

$$\begin{aligned}
\chi_i &= [\tau_i x + \tau_i^3 - 3\lambda\tau_i^2 + \left(3\lambda^3 - 3g_1^2 - 3g_2^2\right)\tau_i + 3\lambda^3 + 3\left(g_1^2 + g_2^2\right)\lambda + 3(f_1 g_1^2 - f_2 g_2^2) \\
&\quad + 3\delta g_1 g_2\left(f_1 + f_2\right)]t,
\end{aligned}$$

τ_i satisfy as

$$\tau_i + \lambda - g_1\frac{g_1 + g_2\delta}{f_1 + \tau_i - \lambda} - g_2\frac{g_2 - g_1\delta}{-f_2 + \tau_i - \lambda} = 0,$$

$$\delta\left(f_1 - f_2 + \tau_i + \lambda\right) - g_2\frac{g_1 + g_2\delta}{f_1 + \tau_i - \lambda} + g_1\frac{g_2 + g_1\delta}{-f_2 + \tau_i - \lambda} = 0. \tag{3.55}$$

With values $g_1 = 1, g_2 = \frac{1}{2}, f_1 = \frac{3}{2}, f_2 = \frac{-3}{2}, \delta = 0$ the seed solution in equation (3.53) changed to

$$A = e^{\frac{3}{8}i(4x - 21t)}, \quad B = \frac{1}{2}e^{-\frac{3}{8}i(4x - 21t)} \tag{3.56}$$

Now the solution in equation (3.54) takes the form as

$$\Psi(\lambda) = \begin{pmatrix} e^{-\frac{16}{3}i(4x - 21t)}\left(\alpha e^P + \beta e^{-P}\right) \\ 0 \\ e^{\frac{16}{3}i(4x - 21t)}\left(\alpha\tau_1 e^P + \beta\tau_2 e^{-P}\right) \\ -\frac{1}{2}e^{\frac{16}{3}i(4x - 21t)}\left(\alpha\tau_1 e^P + \beta\tau_2 e^{-P}\right) \end{pmatrix} \tag{3.57}$$

Here

$$\begin{aligned}
\tau_1 &= \frac{4i}{3 + \sqrt{16\lambda^2 - 24\lambda + 29} - 4\lambda}, \\
\tau_2 &= \frac{-4i}{-3 + \sqrt{16\lambda^2 - 24\lambda + 29} + 4\lambda}, \\
P &= \frac{1}{4}\sqrt{16\lambda^2 - 24\lambda + 29}\,i\left[x + \left(4\lambda^2 + 3\lambda - \frac{1}{4}\right)t\right]. \tag{3.58}
\end{aligned}$$

Suppose $\lambda_1 = \frac{3}{4} - \frac{\sqrt{5}}{2}i$ and expanding vector $\Psi\left(\lambda_1 + \nu^2\right)$ in (3.57) with $\nu = 0$ we get generalized DT as

$$\Psi = \Psi^{[0]} + \Psi^{[1]}\nu^2 + \Psi^{[2]}\nu^4 + \dots\dots\dots + \Psi^{[N]}\nu^{2N} + \dots\dots, \tag{3.59}$$

Here $\Psi^{[i]} = \frac{1}{(2i)!}\frac{\partial^{(2i)}}{\partial\nu^{(2i)}}|_{\nu\to 0}\Psi(\nu), \quad (i = 0, 1, 2, \dots)$.

By the following limit process

$$\Psi[1] = \lim_{\nu\to 0}\frac{B_{\Lambda,\sigma}|_{\lambda=\lambda_1 + \nu^2}\Psi}{\nu^2} + B_{\Lambda,\sigma}|_{\lambda=\lambda_1}\Psi[1] + \frac{\Lambda\Upsilon(\Lambda, \Lambda)^{-1}\Upsilon(\Psi[0], \Lambda)}{\lambda_1 - \lambda_1^*} \tag{3.60}$$

With $\lambda_1 = \frac{3}{4} - \frac{\sqrt{5}}{2}i$ and generalized DT inserting these values into (3.27) we obtain vector rogue waves solution of coupled (cmKdV) of first order with $\alpha = 1, \beta = 0$, as

$$
\begin{aligned}
A[1] &= e^{\frac{3}{8}i(4x-21t)}\left(1 - \frac{-1440it + 64}{Q}\right), \\
B[1] &= \frac{1}{2}e^{-\frac{3}{8}i(4x-21t)}\left(1 - \frac{-1440it + 64}{Q}\right).
\end{aligned}
\tag{3.61}
$$

Here

$$Q = 8145t^2 - 24\sqrt{5}t - 120xt + 80x^2 + 32\sqrt{5}x + 32.$$
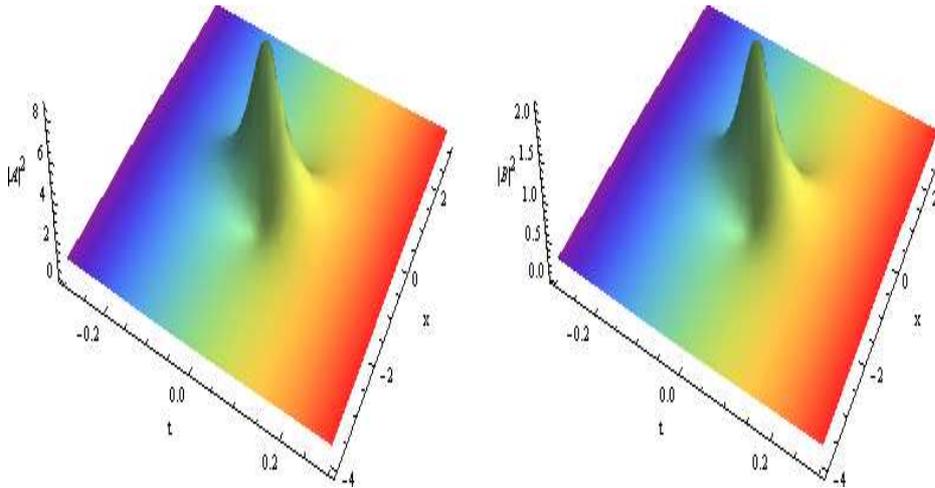
Figure:7 First-order rouge waves having parameters $\alpha = g_1 = 1, g_2 = \frac{1}{2}, \beta = \delta = 0, f_1 = -f_2 = \frac{3}{2}$

Any one can be obtain higher-order solitons with similar fashion.

4. CONCLUDING REMARKS

In this research work, we have obtained various types of solutions of famous coupled (cmKdV) integrable system. For obtaining solutions, we have used binary DT technique. We have obtained zero seed and non-zero seed solutions. Their classification involved, breather solution and one soliton solution in first case, while in non-zero seed solutions, we obtained bright-bright, w-shaped, bright-dark, periodic and rogue waves solutions. In section 3, we have obtained only one soliton solutions when $N = 1$ and rogue waves having order one. We shall try to obtain higher-order soliton solutions and rogue waves in next publications.

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