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Solutions of Some Nielsen-type Integrals Through Hypergeometric Technique with Applications

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Abstract.: Nielsen integrals and similar type of integrals were evaluated by the use of suitable contour integrals and Cauchy's residue theorem. In this article, we obtain the solutions of Nielsen-type integrals and the associated integrals with suitable convergence conditions through hypergeometric approach using Gauss' classical summation theorem and certain trigonometric relations. Some applications of Nielsen-type integrals are also obtained in the form of Weber-Anger type functions.

AMS (MOS) Subject Classification Codes: 33E20; 33B15; 33B10; 33C05; 33E50.
Key Words: Hypergeometric functions; Nielsen Integrals; Gauss' classical summation theorem; Weber function; Anger function; Srivastava-Daoust double hypergeometric function.

1. Introduction and preliminaries

In this paper, we shall use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \cdots\}\,; \mathbb{N}_0 := \mathbb{N} \bigcup \{0\}\,; \mathbb{Z}_0^- := \mathbb{Z}^- \bigcup \{0\} = \{0, -1, -2, -3, \cdots\}\,.$
The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.
The classical Pochhammer symbol $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ is defined by([14, p.22, Eq.(1), p.32, Q.N.(8) and Q.N.(9), see also [18] p.23, Eq.(22) and Eq.(23)]).
A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters [18, p.42, Eq.(1)].
Hypergeometric forms of some elementary functions [18, p.44, Eq.(8),[12], p.489, Entry (7.3.5.1)] are given by:

$$(1-z)^{-a} = {}_1F_0\left[\begin{array}{c} a\,; \\ \quad z \\ -; \end{array}\right],\ |z|<1, a\in\mathbb{C}\backslash\mathbb{Z}_0^-, \tag{1. 1}$$

when $|z|=1$ and $z\neq 1$, then ${}_1F_0\left[\begin{array}{c} a\,; \\ \quad z \\ -; \end{array}\right]$, is well defined when $\mathfrak{Re}(a)<1$.

$$-1+e^{i(\alpha+\beta+\gamma)\pi} = 2\ \sin\left\{(\alpha+\beta+\gamma)\frac{\pi}{2}\right\}\ e^{i(1+\alpha+\beta+\gamma)\frac{\pi}{2}}. \tag{1. 2}$$

Gauss' classical summation theorem[14, p.49, Th.(18)] is given by:

$${}_2F_1\left[\begin{array}{c} \alpha,\ \beta; \\ \quad 1 \\ \gamma; \end{array}\right] = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \tag{1. 3}$$

where $\mathfrak{Re}(\gamma-\alpha-\beta)>0$ and $\gamma\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

$$\frac{1}{\Gamma(d)}\ {}_2F_1\left[\begin{array}{c} a,\ b; \\ \quad z \\ d; \end{array}\right], \tag{1. 4}$$

is well defined for all real and complex values of d.

In article [15], Srivastava and Daoust defined a generalization of the Kampé de Fériet function[2, p.150] by means of the double hypergeometric series (see also[16, p.199] and [17])

$$F^{A:\ B;\ B'}_{C:\ D;\ D'}\left(\begin{array}{c} [(a_A)\ :\ \vartheta,\ \varphi]:[(b_B):\psi];[(b'_{B'}):\psi']; \\ \qquad x,\ y \\ [(c_C)\ :\ \delta,\ \varepsilon]:[(d_D):\eta];[(d'_{D'}):\eta']; \end{array}\right) =$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m\vartheta_j+n\varphi_j}\prod_{j=1}^{B}(b_j)_{m\psi_j}\prod_{j=1}^{B'}(b'_j)_{n\psi'_j}}{\prod_{j=1}^{C}(c_j)_{m\delta_j+n\varepsilon_j}\prod_{j=1}^{D}(d_j)_{m\eta_j}\prod_{j=1}^{D'}(d'_j)_{n\eta'_j}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1. 5}$$

where the coefficients

$$\left\{\begin{array}{c} \vartheta_1,...,\vartheta_A;\ \varphi_1,...,\varphi_A;\ \psi_1,...,\psi_B;\ \psi'_1,...,\psi'_{B'};\ \delta_1,...,\delta_C; \\ \varepsilon_1,...,\varepsilon_C;\ \eta_1,...,\eta_D;\ \eta'_1,...,\eta'_{D'} \end{array}\right. \tag{1. 6}$$

are real and positive.

Indeed it is easy to observe that when $y\to 0$, the double series (1. 5) reduces to the generalized hypergeometric series ${}_p\Psi_q^*$ introduced by Wright[20, 21] and when the positive real coefficients in (1. 6) are all taken as unity, it would equal

$$F^{A:\ B;\ B'}_{C:\ D;\ D'}\left[\begin{array}{c} (a_A):(b_B);(b'_{B'}); \\ \qquad x,\ y \\ (c_C):(d_D);(d'_{D'}); \end{array}\right],$$

where $F^{A:\ B;\ B'}_{C:\ D;\ D'}[x,y]$ denotes Kampé de Fériet's double hypergeometric function in the contracted notation of Burchnall and Chaundy [4, p.112] in preference, for the sake of generality and elegance, to the one used by Kampé de Fériet himself[2, p.150].

$$E_1 = \left(\mu_1^{1+\sum_{j=1}^{D}\eta_j-\sum_{j=1}^{B}\psi_j}\right)\frac{\prod_{j=1}^{C}(\mu_1\delta_j+\mu_2\varepsilon_j)^{\delta_j}\ \prod_{j=1}^{D}(\eta_j)^{\eta_j}}{\prod_{j=1}^{A}(\mu_1\vartheta_j+\mu_2\varphi_j)^{\vartheta_j}\ \prod_{j=1}^{B}(\psi_j)^{\psi_j}}, \tag{1. 7}$$

$$E_2 = \left(\mu_2^{1+\sum_{j=1}^{D'}\eta'_j-\sum_{j=1}^{B'}\psi'_j}\right)\frac{\prod_{j=1}^{C}(\mu_1\delta_j+\mu_2\varepsilon_j)^{\varepsilon_j}\ \prod_{j=1}^{D'}(\eta'_j)^{\eta'_j}}{\prod_{j=1}^{A}(\mu_1\vartheta_j+\mu_2\varphi_j)^{\varphi_j}\ \prod_{j=1}^{B'}(\psi'_j)^{\psi'_j}}. \tag{1. 8}$$

$$\Delta_1 = 1+\sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\eta_j-\sum_{j=1}^{A}\vartheta_j-\sum_{j=1}^{B}\psi_j, \tag{1. 9}$$

$$\Delta_2 = 1+\sum_{j=1}^{C}\varepsilon_j+\sum_{j=1}^{D'}\eta'_j-\sum_{j=1}^{A}\varphi_j-\sum_{j=1}^{B'}\psi'_j. \tag{1. 10}$$

Case I. The double power series in (1. 5) converges for all complex values of x and y when $\Delta_1 > 0$ and $\Delta_2 > 0$.
Case II. The double power series in (1. 5) is convergent when $\Delta_1 = 0,\ \Delta_2 = 0,\ |x| < \rho_1,\ |y| < \rho_2$ where

$$\rho_1 = \min_{\mu_1,\mu_2>0}(E_1),\ \rho_2 = \min_{\mu_1,\mu_2>0}(E_2).$$

Case III. The double power series in (1. 5) would diverge except when, trivially, $x = y = 0$ when $\Delta_1 < 0$ and $\Delta_2 < 0$.
Cauchy's double series identities[18, p.100, Eq.(1), Eq.(2)] are given by:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(m,n) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\Psi(m-n,n), \tag{1. 11}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\Xi(m,n) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Xi(m+n,n), \tag{1. 12}$$

provided that the associated double series on both sides are absolutely convergent.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\Phi(n,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Phi(n+k,k)+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Phi(n+k+1,n+2k+2), \tag{1. 13}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{2n+1}\Phi(n,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Phi(n+k,k)+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Phi(n+k,n+2k+1), \tag{1. 14}$$

provided that the associated double series on both sides are absolutely convergent.
The Weber function $\mathbf{E}_\nu(z)$[19, [3], p.15, Eq.(1)] (also known as Lommel-Weber function given by H.F. Weber in the year 1879) is defined as:

$$\mathbf{E}_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin\,(\nu\theta - z\ \sin\theta)\ d\theta$$

and the Anger function $\mathbf{J}_\nu(z)$([3, p.15, Eq.(2)], see also [1], given by C.T. Anger in the year 1855) is defined as:

$$\mathbf{J}_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos\,(\nu\theta - z\ \sin\theta)\ d\theta.$$

The present article is organized as follows. In Section 2, we have mentioned some Nielsen-type integrals. In Section 3, we have given the solutions of the Nielsen-type integrals. In section 4, we have obtained other associated integrals as the special cases of the main integrals and section 5 is related to the applications of Nielsen-type integrals in the form of weber-Anger type functions.

2. Some Nielsen-type Integrals

When the values of numerator, denominator parameters and arguments leading to the results which do not make sense, are tacitly excluded and the values of parameters and arguments are adjusted in such a way that Gamma functions in the right hand sides are meaningful and well defined. Then

$$\int_0^{\pi}(\sin\, t)^{\alpha}\,(\cos\, t)^{\beta}\, e^{i\gamma t}\,dt = \frac{2^{-\alpha-\beta}\,\pi\,\Gamma(1+\alpha)\,e^{i(\beta+\gamma)\frac{\pi}{2}}}{\Gamma\left(\frac{2+\alpha+\beta+\gamma}{2}\right)\,\Gamma\left(\frac{2+\alpha-\beta-\gamma}{2}\right)}\,{}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\alpha-\beta-\gamma}{2};\\ \frac{2+\alpha-\beta-\gamma}{2}; \end{array} -1\right], \tag{2. 15}$$

where $\mathfrak{Re}(\alpha) > -1,\ \mathfrak{Re}(\beta) > -1,\ \mathfrak{Re}(\alpha+\beta) > -2,\ \frac{2-\alpha-\beta-\gamma}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

$$\int_0^{\pi}(\sin\, t)^{\alpha}\,(\cos\, t)^{\beta}\,\cos\gamma t\,dt = \frac{2^{-\alpha-\beta}\,\pi\,\Gamma(1+\alpha)\,\cos\left\{(\beta+\gamma)\frac{\pi}{2}\right\}}{\Gamma\left(\frac{2+\alpha+\beta+\gamma}{2}\right)\,\Gamma\left(\frac{2+\alpha-\beta-\gamma}{2}\right)}\,{}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\alpha-\beta-\gamma}{2};\\ \frac{2+\alpha-\beta-\gamma}{2}; \end{array} -1\right], \tag{2. 16}$$

where α,β,γ are real numbers such that each Gamma function in the right hand side of equation (2. 16) is well defined and $\alpha > -1,\ \beta > -1,\ (\alpha+\beta) > -2$.

$$\int_0^{\pi}(\sin\, t)^{\alpha}\,(\cos\, t)^{\beta}\,\sin\gamma t\,dt = \frac{2^{-\alpha-\beta}\,\pi\,\Gamma(1+\alpha)\,\sin\left\{(\beta+\gamma)\frac{\pi}{2}\right\}}{\Gamma\left(\frac{2+\alpha+\beta+\gamma}{2}\right)\,\Gamma\left(\frac{2+\alpha-\beta-\gamma}{2}\right)}\,{}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\alpha-\beta-\gamma}{2};\\ \frac{2+\alpha-\beta-\gamma}{2}; \end{array} -1\right], \tag{2. 17}$$

where α,β,γ are real numbers such that each Gamma function in the right hand side of equation (2. 17) is well defined and $\alpha > -1,\ \beta > -1,\ (\alpha+\beta) > -2$.

$$\int_0^{\frac{3\pi}{2}}(\sin\, t)^{\alpha}\,(\cos\, t)^{\beta}\, e^{i\gamma t}\,dt = \frac{2^{-\alpha-\beta}\,\Gamma\left(\frac{2-\alpha-\beta-\gamma}{2}\right)}{(\alpha+\beta+\gamma)}\left\{\frac{e^{i(2\alpha+3\beta+3\gamma-1)\frac{\pi}{2}}\,\Gamma(1+\beta)}{\Gamma\left(\frac{2-\alpha+\beta-\gamma}{2}\right)}\times\right.$$
$$\left.\times{}_2F_1\left[\begin{array}{c} -\alpha,\ \frac{-\alpha-\beta-\gamma}{2};\\ \frac{2-\alpha+\beta-\gamma}{2}; \end{array} -1\right] - \frac{e^{-i(1+\alpha)\frac{\pi}{2}}\,\Gamma(1+\alpha)}{\Gamma\left(\frac{2+\alpha-\beta-\gamma}{2}\right)}\,{}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\alpha-\beta-\gamma}{2};\\ \frac{2+\alpha-\beta-\gamma}{2}; \end{array} -1\right]\right\}, \tag{2. 18}$$

where $\mathfrak{Re}(\alpha) > -1,\ \mathfrak{Re}(\beta) > -1,\ \mathfrak{Re}(\alpha+\beta) > -2,\ \frac{2-\alpha-\beta-\gamma}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

$$\int_0^{\frac{3\pi}{2}}(\sin\, t)^{\alpha}\,(\cos\, t)^{\beta}\,\cos(\gamma t)\,dt = \frac{2^{-\alpha-\beta}\,\Gamma\left(\frac{2-\alpha-\beta-\gamma}{2}\right)}{(\alpha+\beta+\gamma)}\left\{\frac{\Gamma(1+\beta)}{\Gamma\left(\frac{2-\alpha+\beta-\gamma}{2}\right)}\cos\left\{(2\alpha+3\beta+3\gamma-1)\frac{\pi}{2}\right\}\times\right.$$

$$\times\ {}_2F_1\left[\begin{array}{c} -\alpha,\ \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2-\alpha+\beta-\gamma}{2}; \end{array} -1\right] - \frac{\Gamma(1+\alpha)\ \cos\left\{(1+\alpha)\frac{\pi}{2}\right\}}{\Gamma\left(\frac{2+\alpha-\beta-\gamma}{2}\right)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2+\alpha-\beta-\gamma}{2}; \end{array} -1\right]\Bigg\}, \tag{2. 19}$$

where α, β, γ are real numbers such that each Gamma function in the right hand side of equation (2. 19) is well defined and $\alpha > -1,\ \beta > -1,\ (\alpha+\beta) > -2,$
$(\alpha+\beta+\gamma) \neq 2, 4, 6, ...$

$$\int_0^{\frac{3\pi}{2}} (\sin\ t)^{\alpha}\ (\cos\ t)^{\beta}\ \sin(\gamma t)\ dt = \frac{2^{-\alpha-\beta}\ \Gamma\left(\frac{2-\alpha-\beta-\gamma}{2}\right)}{(\alpha+\beta+\gamma)} \left\{ \frac{\Gamma(1+\beta)}{\Gamma\left(\frac{2-\alpha+\beta-\gamma}{2}\right)} \sin\left\{(2\alpha+3\beta+3\gamma-1)\frac{\pi}{2}\right\} \times \right.$$

$$\times\ {}_2F_1\left[\begin{array}{c} -\alpha,\ \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2-\alpha+\beta-\gamma}{2}; \end{array} -1\right] + \frac{\Gamma(1+\alpha)\ \sin\left\{(1+\alpha)\frac{\pi}{2}\right\}}{\Gamma\left(\frac{2+\alpha-\beta-\gamma}{2}\right)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2+\alpha-\beta-\gamma}{2}; \end{array} -1\right]\Bigg\}, \tag{2. 20}$$

where α, β, γ are real numbers such that each Gamma function in the right hand side of equation (2. 20) is well defined and $\alpha > -1,\ \beta > -1,\ (\alpha+\beta) > -2,$
$(\alpha+\beta+\gamma) \neq 2, 4, 6, ...$

$$\int_0^{2\pi} (\sin\ t)^{\alpha}\ (\cos\ t)^{\beta}\ e^{i\gamma t}\ dt = \frac{2^{1-\alpha-\beta}\ \pi\ e^{i(\alpha+2\beta+2\gamma)\frac{\pi}{2}}\ \Gamma(1+\alpha)\ \Gamma\left(\frac{2-\alpha-\beta-\gamma}{2}\right)}{\Gamma(1+\alpha+\beta+\gamma)\ \Gamma(1-\alpha-\beta-\gamma)\ \Gamma\left(\frac{2+\alpha-\beta-\gamma}{2}\right)} \times$$

$$\times\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2+\alpha-\beta-\gamma}{2}; \end{array} -1\right], \tag{2. 21}$$

where $\mathfrak{Re}(\alpha) > -1,\ \mathfrak{Re}(\beta) > -1,\ \mathfrak{Re}(\alpha+\beta) > -2,\ \frac{2-\alpha-\beta-\gamma}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

$$\int_0^{2\pi} (\sin\ t)^{\alpha}\ (\cos\ t)^{\beta}\ \cos(\gamma t)\ dt = \frac{2^{1-\alpha-\beta}\ \pi\ \cos\left\{(\alpha+2\beta+2\gamma)\frac{\pi}{2}\right\}\ \Gamma(1+\alpha)}{\Gamma(1+\alpha+\beta+\gamma)\ \Gamma(1-\alpha-\beta-\gamma)} \times$$

$$\times \frac{\Gamma\left(\frac{2-\alpha-\beta-\gamma}{2}\right)}{\Gamma\left(\frac{2+\alpha-\beta-\gamma}{2}\right)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2+\alpha-\beta-\gamma}{2}; \end{array} -1\right], \tag{2. 22}$$

where α, β, γ are real numbers such that each Gamma function in the right hand side of equation (2. 22) is well defined and $\alpha > -1,\ \beta > -1,\ (\alpha+\beta) > -2,$
$(\alpha+\beta+\gamma) \neq 2, 4, 6, ...$

$$\int_0^{2\pi} (\sin\ t)^{\alpha}\ (\cos\ t)^{\beta}\ \sin(\gamma t)\ dt = \frac{2^{1-\alpha-\beta}\ \pi\ \sin\left\{(\alpha+2\beta+2\gamma)\frac{\pi}{2}\right\}\ \Gamma(1+\alpha)}{\Gamma(1+\alpha+\beta+\gamma)\ \Gamma(1-\alpha-\beta-\gamma)} \times$$

$$\times \frac{\Gamma\left(\frac{2-\alpha-\beta-\gamma}{2}\right)}{\Gamma\left(\frac{2+\alpha-\beta-\gamma}{2}\right)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2+\alpha-\beta-\gamma}{2}; \end{array} -1\right], \tag{2. 23}$$

where α, β, γ are real numbers such that each Gamma function in the right hand side of equation (2. 23) is well defined and $\alpha > -1,\ \beta > -1,\ (\alpha+\beta) > -2,$

$(\alpha+\beta+\gamma)\neq 2,4,6,....$

3. Proof of some Nielsen-type Integrals

Proof of integrals (2. 15), (2. 16) and (2. 17)

Let

$$I_1=\int_0^{\pi}(\sin\ t)^{\alpha}\ (\cos\ t)^{\beta}\ e^{i\gamma t}\ dt=\int_0^{\pi}\left(\frac{e^{it}-e^{-it}}{2i}\right)^{\alpha}\left(\frac{e^{it}+e^{-it}}{2}\right)^{\beta}\ e^{i\gamma t}\ dt$$

$$=\frac{1}{2^{\alpha+\beta}i^{\alpha}}\int_0^{\pi}e^{i(\alpha+\beta+\gamma)t}\ (1-e^{-2it})^{\alpha}(1+e^{-2it})^{\beta}\ dt$$

$$=\frac{1}{2^{\alpha+\beta}i^{\alpha}}\int_0^{\pi}e^{i(\alpha+\beta+\gamma)t}\ {}_1F_0\left[\begin{matrix}-\alpha; & \\ & e^{-2it}\\ -\ ; & \end{matrix}\right]\ {}_1F_0\left[\begin{matrix}-\beta; & \\ & -e^{-2it}\\ -\ ; & \end{matrix}\right]dt$$

Since $\pm e^{-i2t}\neq 1$, but $|\pm e^{-i2t}|=1$, therefore $\mathfrak{Re}(\alpha)>-1$, and $\mathfrak{Re}(\beta)>-1$.

Therefore

$$I_1=\frac{1}{2^{\alpha+\beta}i^{\alpha}}\int_0^{\pi}e^{i(\alpha+\beta+\gamma)t}\sum_{m=0}^{\infty}\frac{(-\alpha)_m\ (e^{-2it})^m}{m!}\sum_{n=0}^{\infty}\frac{(-\beta)_n\ (-1)^n\ (e^{-2it})^n}{n!}dt$$

$$=\frac{1}{2^{\alpha+\beta}i^{\alpha}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\alpha)_m\ (-\beta)_n\ (-1)^n}{m!\ n!}\int_0^{\pi}e^{i(\alpha+\beta+\gamma-2m-2n)t}\ dt$$

$$=\frac{1}{2^{\alpha+\beta}i^{\alpha}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\alpha)_m\ (-\beta)_n\ (-1)^n\ \left\{e^{i(\alpha+\beta+\gamma-2m-2n)\pi}-1\right\}}{i\ (\alpha+\beta+\gamma-2m-2n)\ m!\ n!}$$

$$=\frac{\left\{e^{i(\alpha+\beta+\gamma)\pi}-1\right\}}{2^{\alpha+\beta}\ i^{\alpha+1}\ (\alpha+\beta+\gamma)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\alpha)_m\ (-\beta)_n\ (-1)^n\ \left(\frac{-\alpha-\beta-\gamma}{2}\right)_n\left(\frac{-\alpha-\beta-\gamma+2n}{2}\right)_m}{\left(\frac{2-\alpha-\beta-\gamma}{2}\right)_n\left(\frac{2-\alpha-\beta-\gamma+2n}{2}\right)_m\ m!\ n!}. \tag{3. 24}$$

Using equation (1. 2) in equation (3. 24), we get

$$I_1=\frac{2\sin\left\{(\alpha+\beta+\gamma)\frac{\pi}{2}\right\}e^{i(1+\alpha+\beta+\gamma)\frac{\pi}{2}}}{2^{\alpha+\beta}\ i^{1+\alpha}\ (\alpha+\beta+\gamma)}\sum_{n=0}^{\infty}\frac{(-\beta)_n\ \left(\frac{-\alpha-\beta-\gamma}{2}\right)_n\ (-1)^n}{\left(\frac{2-\alpha-\beta-\gamma}{2}\right)_n\ n!}\sum_{m=0}^{\infty}\frac{(-\alpha)_m\ \left(\frac{-\alpha-\beta-\gamma+2n}{2}\right)_m}{\left(\frac{2-\alpha-\beta-\gamma+2n}{2}\right)_m\ m!}.$$

$$=\frac{2\ \pi\ e^{i(\beta+\gamma)\frac{\pi}{2}}}{2^{\alpha+\beta}\ \Gamma\left(\frac{\alpha+\beta+\gamma}{2}\right)\Gamma\left(\frac{2-\alpha-\beta-\gamma}{2}\right)\ (\alpha+\beta+\gamma)}\sum_{n=0}^{\infty}\frac{(-\beta)_n\ \left(\frac{-\alpha-\beta-\gamma}{2}\right)_n\ (-1)^n}{\left(\frac{2-\alpha-\beta-\gamma}{2}\right)_n\ n!}\times$$

$$\times\ {}_2F_1\left[\begin{matrix}-\alpha,\ \frac{-\alpha-\beta-\gamma+2n}{2}; & \\ & 1\\ \frac{2-\alpha-\beta-\gamma+2n}{2}; & \end{matrix}\right], \tag{3. 25}$$

when $\mathfrak{R}(\alpha)>-1,\ \mathfrak{R}(\beta)>-1,\ \frac{2-\alpha-\beta-\gamma}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-$,
then applying Gauss' classical summation theorem (1. 3) in equation (3. 25), we get the result (2. 15).

When α, β and γ are purely real numbers in equation (2. 15), then we can equate real and imaginary parts

$$\int_0^{\pi} (\sin\ t)^{\alpha}\ (\cos\ t)^{\beta}\ \{\cos(\gamma t) + i\sin(\gamma t)\}\ dt =$$

$$= \frac{2^{-\alpha-\beta}\ \pi\, \Gamma(1+\alpha)\ \left[\cos\left\{(\beta+\gamma)\frac{\pi}{2}\right\} + i\sin\left\{(\beta+\gamma)\frac{\pi}{2}\right\}\right]}{\Gamma\left(\frac{2+\alpha+\beta+\gamma}{2}\right)\ \Gamma\left(\frac{2+\alpha-\beta-\gamma}{2}\right)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2+\alpha-\beta-\gamma}{2}; \end{array} -1\right], \tag{3. 26}$$

where $\alpha > -1,\ \beta > -1,\ (\alpha+\beta) > -2.$
On equating the real and imaginary parts of (3. 26), we get the results (2. 16) and (2. 17).

Proof of integrals (2. 18) to (2. 23)
The solutions of the integrals (2. 18) to (2. 23) can be evaluated by following the same procedure as that of the above integrals. So we omit the details here.

4. Other associated integrals as special cases

In equation (2. 15), put $\beta = 0$, we get Nielsen integral ([10], p.159, Eq.(10), see also [5, p.12, Eq.(29), [7], p.138, Eq.(19(a)), [8], p.15, Eq.(2.6(4)), [11], [9], p.8, last integral]).

$$\int_0^{\pi} (\sin\ t)^{\alpha}\ e^{i\gamma t}\ dt = \frac{2^{-\alpha}\ \pi\ e^{\frac{i\gamma\pi}{2}}\ \Gamma(1+\alpha)}{\Gamma\left(\frac{2+\alpha+\gamma}{2}\right)\ \Gamma\left(\frac{2+\alpha-\gamma}{2}\right)}, \tag{4. 27}$$

where $\mathfrak{Re}(\alpha) > -1$ and $\frac{2-\alpha-\gamma}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
In equation (2. 16), put $\beta = 0$, we get Nielsen integral ([10, p.159, Eq.(8)], see also [7, p.108, Eq.(9(b))])

$$\int_0^{\pi} (\sin\ t)^{\alpha} \cos\ (\gamma t)\ dt = \frac{2^{-\alpha}\ \pi\, \Gamma(1+\alpha)\ \cos\ \left(\frac{\pi}{2}\gamma\right)}{\Gamma\left(\frac{2+\alpha+\gamma}{2}\right)\ \Gamma\left(\frac{2+\alpha-\gamma}{2}\right)},\ \alpha > -1. \tag{4. 28}$$

In equation (2. 17), put $\beta = 0$, we get Nielsen integral ([10, p.159, Eq.(9)], see also [7, p.108, Eq.(9(a))])

$$\int_0^{\pi} (\sin\ t)^{\alpha} \sin\ (\gamma t)\ dt = \frac{2^{-\alpha}\ \pi\, \Gamma(1+\alpha)\ \sin\ \left(\frac{\pi}{2}\gamma\right)}{\Gamma\left(\frac{2+\alpha+\gamma}{2}\right)\ \Gamma\left(\frac{2+\alpha-\gamma}{2}\right)},\ \alpha > -1, \tag{4. 29}$$

where α, γ are real numbers such that each Gamma function in the right hand side of equations (4. 28) and (4. 29) is well defined.
In equation (2. 15), put $\alpha = 0$, we get

$$\int_0^{\pi} (\cos\ t)^{\beta}\ e^{i\gamma t}\ dt = \frac{2^{-\beta}\ \pi\ e^{i(\beta+\gamma)\frac{\pi}{2}}}{\Gamma\left(\frac{2+\beta+\gamma}{2}\right)\ \Gamma\left(\frac{2-\beta-\gamma}{2}\right)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\beta-\gamma}{2}; \\ \frac{2-\beta-\gamma}{2}; \end{array} -1\right], \tag{4. 30}$$

where $\mathfrak{Re}(\beta) > -1$.
In equations (2. 16) and (2. 17), put $\alpha = 0$, we get

$$\int_0^{\pi} (\cos\ t)^{\beta} \cos(\gamma t)\, dt = \frac{2^{-\beta}\ \pi\ \cos\left\{(\beta+\gamma)\frac{\pi}{2}\right\}}{\Gamma\left(\frac{2+\beta+\gamma}{2}\right)\ \Gamma\left(\frac{2-\beta-\gamma}{2}\right)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\beta-\gamma}{2}; \\ \frac{2-\beta-\gamma}{2}; \end{array} -1\right];\ \beta > -1, \tag{4. 31}$$

$$\int_0^{\pi} (\cos\ t)^{\beta} \sin(\gamma t)\, dt = \frac{2^{-\beta}\ \pi\ \sin\left\{(\beta+\gamma)\frac{\pi}{2}\right\}}{\Gamma\left(\frac{2+\beta+\gamma}{2}\right)\ \Gamma\left(\frac{2-\beta-\gamma}{2}\right)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\beta-\gamma}{2}; \\ \frac{2-\beta-\gamma}{2}; \end{array} -1\right];\ \beta > -1, \tag{4. 32}$$

where β and γ are real numbers such that each Gamma function in the right hand sides of equations (4. 31) and (4. 32) is well defined.
In equations (2. 18) and , put where $\beta = 0$, we get

$$\int_0^{\frac{3\pi}{2}} (\sin\ t)^{\alpha}\ e^{i\gamma t}\ dt = \frac{-2^{-\alpha}\ e^{-i(\alpha+1)\frac{\pi}{2}}\ \Gamma\left(\frac{2-\alpha-\gamma}{2}\right)\ \Gamma(1+\alpha)}{(\alpha+\gamma)\ \Gamma\left(\frac{2+\alpha-\gamma}{2}\right)} +$$

$$+\frac{2^{-\alpha}\ e^{i(2\alpha+3\gamma-1)\frac{\pi}{2}}}{(\alpha+\gamma)}\ {}_2F_1\left[\begin{array}{c} -\alpha,\ \frac{-\alpha-\gamma}{2}; \\ \frac{2-\alpha-\gamma}{2}; \end{array} -1\right], \tag{4. 33}$$

where $\mathfrak{Re}(\alpha) > -1$ and $\frac{2-\alpha-\gamma}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
In equations (2. 19) and (2. 20), put $\beta = 0$, we get

$$\int_0^{\frac{3\pi}{2}} (\sin\ t)^{\alpha} \cos(\gamma t)\ dt = \frac{-2^{-\alpha}\ \cos\left\{(1+\alpha)\frac{\pi}{2}\right\}\ \Gamma(1+\alpha)\ \Gamma\left(\frac{2-\alpha-\gamma}{2}\right)}{(\alpha+\gamma)\ \Gamma\left(\frac{2+\alpha-\gamma}{2}\right)} +$$

$$+\frac{2^{-\alpha}\ \cos\left\{(2\alpha+3\gamma-1)\frac{\pi}{2}\right\}}{(\alpha+\gamma)}\ {}_2F_1\left[\begin{array}{c} -\alpha,\ \frac{-\alpha-\gamma}{2}; \\ \frac{2-\alpha-\gamma}{2}; \end{array} -1\right];\ \alpha > -1,\ (\alpha+\gamma) \neq 2, 4, 6, ..., \tag{4. 34}$$

$$\int_0^{\frac{3\pi}{2}} (\sin\ t)^{\alpha} \sin(\gamma t)\ dt = \frac{2^{-\alpha}\ \sin\left\{(1+\alpha)\frac{\pi}{2}\right\}\ \Gamma(1+\alpha)\ \Gamma\left(\frac{2-\alpha-\gamma}{2}\right)}{(\alpha+\gamma)\ \Gamma\left(\frac{2+\alpha-\gamma}{2}\right)} +$$

$$+\frac{2^{-\alpha}\ \sin\left\{(2\alpha+3\gamma-1)\frac{\pi}{2}\right\}}{(\alpha+\gamma)}\ {}_2F_1\left[\begin{array}{c} -\alpha,\ \frac{-\alpha-\gamma}{2}; \\ \frac{2-\alpha-\gamma}{2}; \end{array} -1\right];\alpha > -1,\ (\alpha+\gamma) \neq 2, 4, 6, ..., \tag{4. 35}$$

where α and γ are real numbers such that each Gamma function in the right hand sides of equations (4. 34) and (4. 35) is well defined.
In equation (2. 18), put $\alpha = 0$, we get

$$\int_0^{\frac{3\pi}{2}} (\cos\ t)^{\beta}\ e^{i\gamma t}\ dt = \frac{2^{-\beta}\ e^{i(3\beta+3\gamma-1)\frac{\pi}{2}}\ \Gamma\left(\frac{2-\beta-\gamma}{2}\right)\ \Gamma(1+\beta)}{(\beta+\gamma)\ \Gamma\left(\frac{2+\beta-\gamma}{2}\right)} - \frac{2^{-\beta}\ e^{-i\frac{\pi}{2}}}{(\beta+\gamma)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\beta-\gamma}{2}; \\ \frac{2-\beta-\gamma}{2}; \end{array} -1\right], \tag{4. 36}$$

where $\mathfrak{Re}(\beta) > -1$ and $\frac{2-\beta-\gamma}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
In equations (2. 19) and (2. 20), put $\alpha = 0$, we get

$$\int_0^{\frac{3\pi}{2}} (\cos\ t)^{\beta}\ \cos(\gamma t)\, dt = \frac{2^{-\beta}\ \cos\left\{(3\beta+3\gamma-1)\frac{\pi}{2}\right\}\ \Gamma(1+\beta)\,\Gamma\left(\frac{2-\beta-\gamma}{2}\right)}{(\beta+\gamma)\,\Gamma\left(\frac{2+\beta-\gamma}{2}\right)}, \tag{4. 37}$$

where $\beta > -1,\ (\beta+\gamma) \neq 2, 4, 6, ...,$

$$\int_0^{\frac{3\pi}{2}} (\cos\ t)^{\beta} \sin(\gamma t)\, dt = \frac{2^{-\beta}\ \sin\left\{(3\beta+3\gamma-1)\frac{\pi}{2}\right\}\ \Gamma(1+\beta)\,\Gamma\left(\frac{2-\beta-\gamma}{2}\right)}{(\beta+\gamma)\,\Gamma\left(\frac{2+\beta-\gamma}{2}\right)} + \\ + \frac{2^{-\beta}}{(\beta+\gamma)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\beta-\gamma}{2}; \\ \frac{2-\beta-\gamma}{2}; \end{array} -1\right], \tag{4. 38}$$

where $\beta > -1,\ (\beta+\gamma) \neq 2, 4, 6, ...$ and β, γ are real numbers such that each Gamma function in the right hand sides of equations (4. 37) and (4. 38) is well defined.
In equation (2. 21), put $\beta = 0$, we get

$$\int_0^{2\pi} (\sin\ t)^{\alpha}\ e^{i\gamma t}\, dt = \frac{2^{1-\alpha}\ \pi\ e^{i(\alpha+2\gamma)\frac{\pi}{2}}\ \Gamma\left(\frac{2-\alpha-\gamma}{2}\right)\ \Gamma(1+\alpha)}{\Gamma(1+\alpha+\gamma)\ \Gamma(1-\alpha-\gamma)\ \Gamma\left(\frac{2+\alpha-\gamma}{2}\right)}, \tag{4. 39}$$

where $\mathfrak{Re}(\alpha) > -1$ and $\frac{2-\alpha-\gamma}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
In equations (2. 22) and (2. 23), put $\beta = 0$, we get

$$\int_0^{2\pi} (\sin\ t)^{\alpha} \cos(\gamma t)\, dt = \frac{2^{1-\alpha}\ \pi\ \cos\left\{(\alpha+2\gamma)\frac{\pi}{2}\right\}\ \Gamma\left(\frac{2-\alpha-\gamma}{2}\right)\ \Gamma(1+\alpha)}{\Gamma(1+\alpha+\gamma)\ \Gamma(1-\alpha-\gamma)\ \Gamma(\frac{2+\alpha-\gamma}{2})}, \tag{4. 40}$$

where $\alpha > -1,\ (\alpha+\gamma) \neq 2, 4, 6, ...,$

$$\int_0^{2\pi} (\sin\ t)^{\alpha} \sin(\gamma t)\, dt = \frac{2^{1-\alpha}\ \pi\ \sin\left\{(\alpha+2\gamma)\frac{\pi}{2}\right\}\ \Gamma\left(\frac{2-\alpha-\gamma}{2}\right)\ \Gamma(1+\alpha)}{\Gamma(1+\alpha+\gamma)\ \Gamma(1-\alpha-\gamma)\ \Gamma(\frac{2+\alpha-\gamma}{2})}, \tag{4. 41}$$

where $\alpha > -1,\ (\alpha+\gamma) \neq 2, 4, 6, ...$ and α, γ are real numbers such that each Gamma function in the right hand sides of equations (4. 40) and (4. 41) is well defined.
In equation (2. 21), put $\alpha = 0$, we get

$$\int_0^{2\pi} (\cos\ t)^{\beta}\ e^{i\gamma t}\, dt = \frac{2^{1-\beta}\ \pi\ e^{i(\beta+\gamma)\pi}}{\Gamma(1+\beta+\gamma)\ \Gamma(1-\beta-\gamma)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\beta-\gamma}{2}; \\ \frac{2-\beta-\gamma}{2}; \end{array} -1\right], \tag{4. 42}$$

where $\mathfrak{Re}(\beta) > -1$ and $\frac{2-\beta-\gamma}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
In equations (2. 22) and (2. 23), put $\alpha = 0$, we get

$$\int_0^{2\pi} (\cos\ t)^{\beta} \cos(\gamma t)\, dt = \frac{2^{1-\beta}\ \pi\ \cos\{(\beta+\gamma)\pi\}}{\Gamma(1+\beta+\gamma)\ \Gamma(1-\beta-\gamma)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\beta-\gamma}{2}; \\ \frac{2-\beta-\gamma}{2}; \end{array} -1\right], \tag{4. 43}$$

where $\beta > -1,\ (\beta+\gamma) \neq 2, 4, 6, ...,$

$$\int_0^{2\pi} (\cos\ t)^\beta \sin\ (\gamma t)\ dt = \frac{2^{1-\beta}\ \pi\ \sin\{(\beta+\gamma)\pi\}}{\Gamma\,(1+\beta+\gamma)\ \Gamma\,(1-\beta-\gamma)}\ {}_2F_1\left[\begin{array}{c} -\beta,\ \frac{-\beta-\gamma}{2};\\ \\ \frac{2-\beta-\gamma}{2}; \end{array}\ -1\right], \tag{4. 44}$$

where $\beta > -1,\ (\beta+\gamma) \neq 2, 4, 6, ...$ and β, γ are real numbers such that each Gamma function in the right hand sides of equations (4. 43) and (4. 44) is well defined.

5. Applications in Weber-Anger type functions

Inspired by the hypergeometric forms of Weber and Anger functions [6, p.948, Eq.(8.581(1)) and Eq.(8.581(2)), [12]], we find the applications of Nielsen-type integrals in the form of Weber-Anger type functions.

Weber-type functions:

$$\mathbf{E}_\nu^{(1)}(z) = \frac{1}{\pi}\int_0^{\frac{3\pi}{2}} \sin\,(\nu\theta - z\ \sin\theta)\ d\theta =$$

$$= \frac{1}{\pi\nu}\ {}_1F_2\left[\begin{array}{c} 1\ ;\\ \\ \frac{2+\nu}{2}, \frac{2-\nu}{2}; \end{array}\ \frac{-z^2}{4}\right] - \frac{z}{\pi(1-\nu^2)}\ {}_1F_2\left[\begin{array}{c} 1\ ;\\ \\ \frac{3+\nu}{2}, \frac{3-\nu}{2}; \end{array}\ \frac{-z^2}{4}\right] -$$

$$-\frac{\cos\left\{(3\nu)\frac{\pi}{2}\right\}}{\pi\nu}\ F^{1:2;0}_{3:1;0}\left(\begin{array}{c} [1:2,2]\ :[\frac{\nu}{2}:1],[1:1];-;\\ \\ [1:2,1],[1:1,1],[\frac{1}{2}:1,1]:\ [\frac{2+\nu}{2}:1];-; \end{array}\ -\frac{z^2}{16}, -\frac{z^2}{16}\right) -$$

$$-\frac{z^2\ \cos\left\{(3\nu)\frac{\pi}{2}\right\}}{8\pi(2-\nu)}\ F^{1:0;2}_{3:0;1}\left(\begin{array}{c} [3:2,2]\ :-;[\frac{2-\nu}{2}:1],[1:1];\\ \\ [3:1,2],[2:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{4-\nu}{2}:1]; \end{array}\ -\frac{z^2}{16}, -\frac{z^2}{16}\right) +$$

$$+\frac{z\ \sin\left\{(3\nu)\frac{\pi}{2}\right\}}{2\pi(1+\nu)}\ F^{1:2;0}_{3:1;0}\left(\begin{array}{c} [2:2,2]\ :[\frac{1+\nu}{2}:1],[1:1];-;\\ \\ [2:2,1],[1:1,1],[\frac{3}{2}:1,1]:\ [\frac{3+\nu}{2}:1];-; \end{array}\ -\frac{z^2}{16}, -\frac{z^2}{16}\right) +$$

$$+\frac{z\,\Gamma(\frac{\upsilon-1}{2})\ \sin\left\{(3\nu)\frac{\pi}{2}\right\}}{4\pi\ \Gamma\left(\frac{1+\nu}{2}\right)}\ F^{1:0;2}_{3:0;1}\left(\begin{array}{c} [2:2,2]\ :-;[\frac{1-\nu}{2}:1],[1:1];\\ \\ [2:1,2],[1:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{3-\nu}{2}:1]; \end{array}\ -\frac{z^2}{16}, -\frac{z^2}{16}\right). \tag{5. 45}$$

$$\mathbf{E}_\nu^{(2)}(z) = \frac{1}{\pi}\int_0^{2\pi} \sin\,(\nu\theta - z\ \sin\theta)\ d\theta = \frac{2\sin\,(\pi\nu)}{\Gamma(1+\nu)\ \Gamma(1-\nu)}\ {}_1F_2\left[\begin{array}{c} 1\ ;\\ \\ \frac{2+\nu}{2}, \frac{2-\nu}{2}; \end{array}\ \frac{-z^2}{4}\right] +$$

$$+\frac{2z\sin\,(\pi\nu)}{(1-\nu)\Gamma(-\nu)\Gamma(2+\nu)}\ {}_1F_2\left[\begin{array}{c} 1\ ;\\ \\ \frac{3+\nu}{2}, \frac{3-\nu}{2}; \end{array}\ \frac{-z^2}{4}\right]. \tag{5. 46}$$

$$\mathbf{E}_\nu^{(3)}(z) = \frac{1}{\pi}\int_0^{\pi} \sin\left(\nu\theta - z\ \cos\theta\right) d\theta =$$

$$= \frac{\sin\left(\frac{\pi}{2}\nu\right)}{\Gamma\left(\frac{2+\nu}{2}\right)\ \Gamma\left(\frac{2-\nu}{2}\right)}\ F^{1:2;0}_{3:1;0}\left(\begin{array}{c} [1:2,2]\ :[\frac{\nu}{2}:1],[1:1];-; \\ [1:2,1],[1:1,1],[\frac{1}{2}:1,1]:\ [\frac{2+\nu}{2}:1];-; \end{array} -\tfrac{z^2}{16},-\tfrac{z^2}{16}\right)+$$

$$+\frac{z^2\sin\left(\frac{\pi}{2}\nu\right)}{4(2-\nu)\,\Gamma\left(\frac{\nu}{2}\right)\ \Gamma\left(\frac{2-\nu}{2}\right)}\ F^{1:0;2}_{3:0;1}\left(\begin{array}{c} [3:2,2]\ :-;[\frac{2-\nu}{2}:1],[1:1]; \\ [3:1,2],[2:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{4-\nu}{2}:1]; \end{array} -\tfrac{z^2}{16},-\tfrac{z^2}{16}\right)+$$

$$+\frac{z\sin\left(\frac{\pi}{2}\nu\right)}{(1+\nu)\,\Gamma\left(\frac{1+\nu}{2}\right)\ \Gamma\left(\frac{1-\nu}{2}\right)}\ F^{1:2;0}_{3:1;0}\left(\begin{array}{c} [2:2,2]\ :[\frac{1+\nu}{2}:1],[1:1];-; \\ [2:2,1],[1:1,1],[\frac{3}{2}:1,1]:\ [\frac{3+\nu}{2}:1];-; \end{array} -\tfrac{z^2}{16},-\tfrac{z^2}{16}\right)+$$

$$+\frac{z\sin\left(\frac{\pi}{2}\nu\right)}{2\,\Gamma\left(\frac{1+\nu}{2}\right)\ \Gamma\left(\frac{1+\nu}{2}\right)}\ F^{1:0;2}_{3:0;1}\left(\begin{array}{c} [2:2,2]\ :-;[\frac{1-\nu}{2}:1],[1:1]; \\ [2:1,2],[1:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{3-\nu}{2}:1]; \end{array} -\tfrac{z^2}{16},-\tfrac{z^2}{16}\right). \qquad (5.\ 47)$$

$$\mathbf{E}_\nu^{(4)}(z) = \frac{1}{\pi}\int_0^{\frac{3\pi}{2}} \sin\left(\nu\theta - z\ \cos\theta\right) d\theta =$$

$$= -\frac{\cos\left\{(3\nu)\frac{\pi}{2}\right\}}{\pi\nu}\ {}_1F_2\left[\begin{array}{c} 1\ ; \\ \frac{2+\nu}{2},\ \frac{2-\nu}{2}; \end{array} \frac{-z^2}{4}\right]+\frac{z\cos\left\{(3\nu)\frac{\pi}{2}\right\}}{\pi(1-\nu^2)}\ {}_1F_2\left[\begin{array}{c} 1\ ; \\ \frac{3+\nu}{2},\ \frac{3-\nu}{2}; \end{array} \frac{-z^2}{4}\right]+$$

$$+\frac{1}{\pi\nu}\ F^{1:2;0}_{3:1;0}\left(\begin{array}{c} [1:2,2]\ :[\frac{\nu}{2}:1],[1:1];-; \\ [1:2,1],[1:1,1],[\frac{1}{2}:1,1]:\ [\frac{2+\nu}{2}:1];-; \end{array} -\tfrac{z^2}{16},-\tfrac{z^2}{16}\right)+$$

$$+\frac{z^2}{8\pi\ (2-\nu)}\ F^{1:0;2}_{3:0;1}\left(\begin{array}{c} [3:2,2]\ :-;[\frac{2-\nu}{2}:1],[1:1]; \\ [3:1,2],[2:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{4-\nu}{2}:1]; \end{array} -\tfrac{z^2}{16},-\tfrac{z^2}{16}\right). \qquad (5.\ 48)$$

$$\mathbf{E}_\nu^{(5)}(z) = \frac{1}{\pi}\int_0^{2\pi} \sin\left(\nu\theta - z\ \cos\theta\right) d\theta =$$

$$= \frac{2\sin\left(\pi\nu\right)}{\Gamma(1+\nu)\ \Gamma(1-\nu)}\ F^{1:2;0}_{3:1;0}\left(\begin{array}{c} [1:2,2]\ :[\frac{\nu}{2}:1],[1:1];-; \\ [1:2,1],[1:1,1],[\frac{1}{2}:1,1]:\ [\frac{2+\nu}{2}:1];-; \end{array} -\tfrac{z^2}{16},-\tfrac{z^2}{16}\right)+$$

$$+\frac{z^2\sin\left(\pi\nu\right)}{8(2-\nu)\Gamma(\nu)\ \Gamma(1-\nu)}\ F^{1:0;2}_{3:0;1}\left(\begin{array}{c} [3:2,2]\ :-;[\frac{2-\nu}{2}:1],[1:1]; \\ [3:1,2],[2:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{4-\nu}{2}:1]; \end{array} -\tfrac{z^2}{16},-\tfrac{z^2}{16}\right)+$$

$$+\frac{z\cos\left(\pi\nu\right)}{\Gamma(-\nu)\Gamma(2+\nu)}\ F^{1:2;0}_{3:1;0}\left(\begin{array}{c} [2:2,2]\ :[\frac{1+\nu}{2}:1],[1:1];-; \\ [2:2,1],[1:1,1],[\frac{3}{2}:1,1]:\ [\frac{3+\nu}{2}:1];-; \end{array} -\tfrac{z^2}{16},-\tfrac{z^2}{16}\right)+$$

$$+\frac{z\,\Gamma\left(\frac{1-\nu}{2}\right)\cos(\pi\nu)}{2\Gamma(-\nu)\Gamma(1+\nu)\Gamma\left(\frac{1+\nu}{2}\right)}\,F^{1:0;2}_{3:0;1}\left(\begin{array}{c}[2:2,2]\ :-;[\frac{1-\nu}{2}:1],[1:1];\\ \\ [2:1,2],[1:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{3-\nu}{2}:1];\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right). \tag{5. 49}$$

Anger-type Functions:

$$\mathbf{J}^{(1)}_{\nu}(z)=\frac{1}{\pi}\int_0^{\frac{3\pi}{2}}\cos(\nu\theta-z\ \sin\theta)\,d\theta=$$

$$=\frac{\sin\left\{(3\nu)\frac{\pi}{2}\right\}}{\pi\nu}\,F^{1:2;0}_{3:1;0}\left(\begin{array}{c}[1:2,2]\ :[\frac{\nu}{2}:1],[1:1];-;\\ \\ [1:2,1],[1:1,1],[\frac{1}{2}:1,1]:\ [\frac{2+\nu}{2}:1];-;\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right)+$$

$$+\frac{z^2\sin\left\{(3\nu)\frac{\pi}{2}\right\}}{8\pi\,(2-\nu)}\,F^{1:0;2}_{3:0;1}\left(\begin{array}{c}[3:2,2]\ :-;[\frac{2-\nu}{2}:1],[1:1];\\ \\ [3:1,2],[2:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{4-\nu}{2}:1];\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right)+$$

$$+\frac{z\ \cos\left\{(3\nu)\frac{\pi}{2}\right\}}{2\pi\,(1+\nu)}\,F^{1:2;0}_{3:1;0}\left(\begin{array}{c}[2:2,2]\ :[\frac{1+\nu}{2}:1],[1:1];-;\\ \\ [2:2,1],[1:1,1],[\frac{3}{2}:1,1]:\ [\frac{3+\nu}{2}:1];-;\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right)+$$

$$+\frac{z\,\Gamma\left(\frac{1-\nu}{2}\right)\ \cos\left\{(3\nu)\frac{\pi}{2}\right\}}{4\pi\ \Gamma\left(\frac{1+\nu}{2}\right)}\,F^{1:0;2}_{3:0;1}\left(\begin{array}{c}[2:2,2]\ :-;[\frac{1-\nu}{2}:1],[1:1];\\ \\ [2:1,2],[1:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{3-\nu}{2}:1];\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right). \tag{5. 50}$$

$$\mathbf{J}^{(2)}_{\nu}(z)=\frac{1}{\pi}\int_0^{2\pi}\cos(\nu\theta-z\ \sin\theta)\,d\theta=\frac{2\cos(\pi\nu)}{\Gamma(1+\nu)\,\Gamma(1-\nu)}\,{}_1F_2\left[\begin{array}{c}1\ ;\\ \\ \frac{2+\nu}{2},\frac{2-\nu}{2};\end{array}\ \frac{-z^2}{4}\right]+$$

$$+\ \frac{2z\cos(\pi\nu)}{(1-\nu)\Gamma(-\nu)\Gamma(2+\nu)}\,{}_1F_2\left[\begin{array}{c}1\ ;\\ \\ \frac{3+\nu}{2},\frac{3-\nu}{2};\end{array}\ \frac{-z^2}{4}\right]. \tag{5. 51}$$

$$\mathbf{J}^{(3)}_{\nu}(z)=\frac{1}{\pi}\int_0^{\pi}\cos(\nu\theta-z\ \cos\theta)\,d\theta=$$

$$=\frac{\cos\left(\nu\frac{\pi}{2}\right)}{\Gamma\left(\frac{2+\nu}{2}\right)\Gamma\left(\frac{2-\nu}{2}\right)}\,F^{1:2;0}_{3:1;0}\left(\begin{array}{c}[1:2,2]\ :[\frac{\nu}{2}:1],[1:1];-;\\ \\ [1:2,1],[1:1,1],[\frac{1}{2}:1,1]:\ [\frac{2+\nu}{2}:1];-;\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right)+$$

$$+\frac{z^2\cos\left(\nu\frac{\pi}{2}\right)}{4(2-\nu)\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{2-\nu}{2}\right)}\,F^{1:0;2}_{3:0;1}\left(\begin{array}{c}[3:2,2]\ :-;[\frac{2-\nu}{2}:1],[1:1];\\ \\ [3:1,2],[2:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{4-\nu}{2}:1];\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right)+$$

$$+\frac{z\ \cos\left(\nu\frac{\pi}{2}\right)}{(1+\nu)\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}\,F^{1:2;0}_{3:1;0}\left(\begin{array}{c}[2:2,2]\ :[\frac{1+\nu}{2}:1],[1:1];-;\\ \\ [2:2,1],[1:1,1],[\frac{3}{2}:1,1]:\ [\frac{3+\nu}{2}:1];-;\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right)+$$

$$+\frac{z\ \cos\left(\nu\frac{\pi}{2}\right)}{2\,\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1+\nu}{2}\right)}\,F^{1:0;2}_{3:0;1}\left(\begin{array}{c}[2:2,2]\ :-;[\frac{1-\nu}{2}:1],[1:1];\\ \\ [2:1,2],[1:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{3-\nu}{2}:1];\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right). \tag{5. 52}$$

$$\mathbf{J}^{(4)}_{\nu}(z)=\frac{1}{\pi}\int_0^{\frac{3\pi}{2}}\cos\left(\nu\theta-z\ \cos\theta\right)d\theta=$$

$$=\frac{\sin\left\{(3\nu)\frac{\pi}{2}\right\}}{\pi\nu}\,{}_1F_2\left[\begin{array}{c}1\ ;\\ \\ \frac{2+\nu}{2},\frac{2-\nu}{2};\end{array}\ \frac{-z^2}{4}\right]-\frac{z\sin\left\{(3\nu)\frac{\pi}{2}\right\}}{\pi(1-\nu^2)}\,{}_1F_2\left[\begin{array}{c}1\ ;\\ \\ \frac{3+\nu}{2},\frac{3-\nu}{2};\end{array}\ \frac{-z^2}{4}\right]+$$

$$+\frac{z}{2\pi\,(1+\nu)}\,F^{1:2;0}_{3:1;0}\left(\begin{array}{c}[2:2,2]\ :[\frac{1+\nu}{2}:1],[1:1];-;\\ \\ [2:2,1],[1:1,1],[\frac{3}{2}:1,1]:\ [\frac{3+\nu}{2}:1];-;\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right)+$$

$$+\frac{z\,\Gamma\left(\frac{1-\nu}{2}\right)}{4\pi\,\Gamma\left(\frac{1+\nu}{2}\right)}\,F^{1:0;2}_{3:0;1}\left(\begin{array}{c}[2:2,2]\ :-;[\frac{1-\nu}{2}:1],[1:1];\\ \\ [2:1,2],[1:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{3-\nu}{2}:1];\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right). \tag{5. 53}$$

$$\mathbf{J}^{(5)}_{\nu}(z)=\frac{1}{\pi}\int_0^{2\pi}\cos\left(\nu\theta-z\ \cos\theta\right)d\theta=$$

$$=\frac{2\cos\left(\pi\nu\right)}{\Gamma(1+\nu)\,\Gamma(1-\nu)}\,F^{1:2;0}_{3:1;0}\left(\begin{array}{c}[1:2,2]\ :[\frac{\nu}{2}:1],[1:1];-;\\ \\ [1:2,1],[1:1,1],[\frac{1}{2}:1,1]:\ [\frac{2+\nu}{2}:1];-;\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right)+$$

$$+\frac{z^2\cos\left(\pi\nu\right)}{4(2-\nu)\Gamma\left(\nu\right)\Gamma\left(1-\nu\right)}\,F^{1:0;2}_{3:0;1}\left(\begin{array}{c}[3:2,2]\ :-;[\frac{2-\nu}{2}:1],[1:1];\\ \\ [3:1,2],[2:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{4-\nu}{2}:1];\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right)-$$

$$-\frac{z\ \sin\left(\pi\nu\right)}{\Gamma\left(2+\nu\right)\Gamma\left(-\nu\right)}\,F^{1:2;0}_{3:1;0}\left(\begin{array}{c}[2:2,2]\ :[\frac{1+\nu}{2}:1],[1:1];-;\\ \\ [2:2,1],[1:1,1],[\frac{3}{2}:1,1]:\ [\frac{3+\nu}{2}:1];-;\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right)-$$

$$-\frac{z\,\Gamma\left(\frac{1-\nu}{2}\right)\ \sin\left(\pi\nu\right)}{2\,\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(1+\nu\right)\Gamma\left(-\nu\right)}\,F^{1:0;2}_{3:0;1}\left(\begin{array}{c}[2:2,2]\ :-;[\frac{1-\nu}{2}:1],[1:1];\\ \\ [2:1,2],[1:1,1],[\frac{3}{2}:1,1]:-;\ [\frac{3-\nu}{2}:1];\end{array}\ -\frac{z^2}{16},-\frac{z^2}{16}\right). \tag{5. 54}$$

Proof of Weber-type functions (5. 45) to (5. 49)

$$\mathbf{E}^{(1)}_{\nu}(z)=\frac{1}{\pi}\int_0^{\frac{3\pi}{2}}\sin\left(\nu\theta-z\ \sin\theta\right)d\theta=$$

$$=\frac{1}{\pi}\left\{\int_0^{\frac{3\pi}{2}}\sin\left(\nu\theta\right)\ \sum_{n=0}^{\infty}\frac{(-1)^n\ (z\ \sin\theta)^{2n}}{(2n)!}\ d\theta-\int_0^{\frac{3\pi}{2}}\cos\left(\nu\theta\right)\ \sum_{n=0}^{\infty}\frac{(-1)^n\ (z\ \sin\theta)^{2n+1}}{(2n+1)!}\ d\theta\right\}. \tag{5. 55}$$

Interchanging the order of summation and integration, we get

$$\mathbf{E}_{\nu}^{(1)}(z)=\frac{1}{\pi}\left\{\sum_{n=0}^{\infty}\frac{(-1)^{n}\ z^{2n}}{(2n)!}\int_{0}^{\frac{3\pi}{2}}(\sin\theta)^{2n}\ \sin(\nu\theta)\ d\theta-\sum_{n=0}^{\infty}\frac{(-1)^{n}\ z^{2n+1}}{(2n+1)!}\int_{0}^{\frac{3\pi}{2}}(\sin\theta)^{2n+1}\ \cos(\nu\theta)\ d\theta\right\}. \quad (5.\ 56)$$

Using equations (4. 34) and (4. 35) in equation (5. 56), we get

$$\mathbf{E}_{\nu}^{(1)}(z)=\frac{1}{\pi\nu}\sum_{n=0}^{\infty}\frac{(-1)^{n}\ z^{2n}}{\left(\frac{2+\nu}{2}\right)_{n}\left(\frac{2-\nu}{2}\right)_{n}2^{2n}}-\frac{z}{\pi(1-\nu)^{2}}\sum_{n=0}^{\infty}\frac{(-1)^{n}\ z^{2n}}{\left(\frac{3+\nu}{2}\right)_{n}\left(\frac{3-\nu}{2}\right)_{n}2^{2n}}+$$
$$+\frac{\sin\left\{(3\nu-1)\frac{\pi}{2}\right\}}{\pi\nu}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{\left(\frac{\nu}{2}\right)_{n}(-2n)_{k}\left(\frac{-2n-\nu}{2}\right)_{k}(-1)^{n+k}\ z^{2n}}{\left(\frac{2+\nu}{2}\right)_{n}\left(\frac{1}{2}\right)_{n}\left(\frac{2-2n-\nu}{2}\right)_{k}4^{2n}\ n!\ k!}-$$
$$-\frac{z\ \cos\left\{(3\nu+1)\frac{\pi}{2}\right\}}{2\pi(1+\nu)}\sum_{n=0}^{\infty}\sum_{k=0}^{2n+1}\frac{\left(\frac{1+\nu}{2}\right)_{n}(-2n-1)_{k}\left(\frac{-2n-1-\nu}{2}\right)_{k}(-1)^{n+k}\ z^{2n}}{\left(\frac{3+\nu}{2}\right)_{n}\left(\frac{3}{2}\right)_{n}\left(\frac{1-2n-\nu}{2}\right)_{k}4^{2n}\ n!\ k!}. \quad (5.\ 57)$$

Using double series identities (1. 13) and (1. 14) in equation (5. 57)and further applying the definitions of generalized hypergeometric series and double hypergeometric function of Srivastava and Daoust (1. 5), we get the result (5. 45).
Similarly, we can evaluate $\mathbf{E}_{\nu}^{(2)}(z), \mathbf{E}_{\nu}^{(3)}(z), \mathbf{E}_{\nu}^{(4)}(z)$ and $\mathbf{E}_{\nu}^{(5)}(z)$. So we omit the details here.

Proof of Anger-type functions (5. 50) to (5. 54)
In order to evaluate $\mathbf{J}_{\nu}^{(1)}(z)$, we follow the same procedure as above and arrive at the result. Similarly, we can evaluate $\mathbf{J}_{\nu}^{(2)}(z), \mathbf{J}_{\nu}^{(3)}(z), \mathbf{J}_{\nu}^{(4)}(z)$ and $\mathbf{J}_{\nu}^{(5)}(z)$. So we omit the details here.

6. Concluding remarks

In this paper, we evaluated Nielsen-type integrals through hypergeometric approach and also deduced some applications of Nielsen-type integrals in the form of Weber-Anger type functions. We conclude this paper with the remark that the results obtained are useful for investigating certain other definite integrals, which may be different from the current integrals, in a similar and logical manner. We may consider the definite integrals whose limit ranges from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, 0 to $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ to π (see Qureshi et al. [13]). Moreover, the results derived in this paper are quite significant, general in character and these are expected to lead more potential applications in several fields of Mathematical, Physical and Engineering Sciences.

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