Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(5)(2019) pp. 43-59

New Quartic B-Spline Approximation for Numerical Solution of Third Order Singular Boundary Value Problems

Muhammad Kashif Iqbal
Department of Mathematics,
National College of Business Administration & Economics, Lahore, Pakistan.
Department of Mathematics,
Government College University, Faisalabad, Pakistan.
Email: kashifiqbal@gcuf.edu.pk

Muhammad Abbas
Department of Mathematics,
University of Sargodha, Sargodha, Pakistan.
Email: muhammad.abbas@uos.edu.pk

Bushra Zafar
Department of Computer Science,
Government College University, Faisalabad, Pakistan.
Email: bushrazafar@gcuf.edu.pk

Received: 04 June, 2018 / Accepted: 09 November, 2018 / Published online: 05 March, 2019

Abstract. This article delves the approximate solution for third order singular boundary value problems using quartic B-spline collocation method furnished with a new approximation for third order derivative. The new approximation technique is tested on a class of third order singular initial/boundary value problems and the computational outcomes are compared with those existing in open literature. Due to its straight forward and simple application, the proposed numerical algorithm provides more accurate results as compared to other variants on the topic.

AMS (MOS) Subject Classification Codes: 34B15; 34B16; 74H15; 65L10; 65L11
Key Words: Singular boundary value problems, Quartic B-spline functions, Spline interpolation, Quasi-linearization technique.

1. Introduction

Consider the following class of third order singular boundary value problem (SBVP)

$$\alpha u'''(x) + \frac{\beta}{x} u''(x) + r(x)u'(x) = f(x, u),\ x \in [0, 1], \tag{1. 1}$$

with one of the following sets of conditions

$$\begin{cases} u(0)=\alpha_1, \;\; u'(0)=\alpha_2, \;\; u(1)=\alpha_3 \\ u(0)=\alpha_1, \;\; u'(1)=\alpha_2, \;\; u(1)=\alpha_3 \\ u(0)=\alpha_1, \;\; u'(0)=\alpha_2, \;\; u'(1)=\alpha_3 \\ u(0)=\alpha_1, \;\; u'(0)=\alpha_2, \;\; u''(0)=0, \end{cases} \tag{1. 2}$$

where β is the shape parameter, α, α_i's are constants and $r(x)$ is a smooth function. To ensure the existence of unique solution to (1. 1), we assume that both f and f_u are continuous with $f_u \geq 0$ in the entire domain. The equation (1. 1) carries singularity at $x = 0$ and non-linear term, which affects the convergence of numerical techniques. Therefore, these types of initial/boundary value problems have always remained attractive for the researchers. In recent years, the numerical treatment of boundary value problems (BVP's) have gained a considerable amount of attention due to their broad scope of applications in real life phenomena such as boundary layer theory, chemical reactions, thermal explosions, fluid dynamics, atomic nuclear reactions and theory of elastic stability [3, 10, 13, 21, 25, 26]. Khuri [16] proposed a new decomposition method for solving generalized Emden-Flower type equations. The power series solution for higher order SBVP's has been proposed in [11, 12, 18] by means of a modified form of Adomian decomposition method (MADM). Aruna and Kanth [4] used Differential transformation method (DTM) in order to explore the series solution for higher order SBVP's. The numerical solution of a class of 3^{rd} order Emden-Flower type equations has been discussed in [23] by means of Adomian decomposition method (ADM). Taiwo and Hassan [22] formulated an Iterative decomposition method (IDM) and a numerical method based on Bernstein polynomials for higher order non-linear SBVP's. Wazwaz [24] proposed Variational iteration method (VIM) for series solution of two new kinds of 3^{rd} order singular initial value problems. Dezhbord *et al.* [7] presented a numerical scheme based on reproducing kernel method (RKM) for solving higher order SBVP's. The use of spline functions for solving BVP's has become very frequent in recent years. These functions possess high degree of smoothness and provide approximate solution in the entire domain with great accuracy. The third and fourth degree B-spline functions have been employed in [1, 5, 6, 8, 9, 14, 15, 17] to study the approximate solution of BVP's. Akram [2] employed the fourth degree splines for numerical solution of 3^{rd} order singularly perturbed BVP's. The QBS functions were utilized in [20] for solving third order singularly perturbed SBVP's. Iqbal *et al.* [13] proposed a new cubic B-spline (CBS) approximation scheme for numerical solution of third order SBVP's.

In this paper, the approximate solution for 3^{rd} order SBVP's is studied. The proposed numerical scheme is based on a new QBS approximation for third order derivative. In recent years, different numerical techniques have been proposed for solving third order SBVP's but as yet as we know, this approximation technique is novel for the said purpose.

This paper is composed as follows: Some basic concepts of fourth degree basis spline interpolation are discussed in section 2. The new approximation to $u'''(x)$ has been formulated in section 3. Section 4 covers the description of numerical method. In order to prove the uniform convergence of the presented numerical algorithm, an error analysis is carried out in section 5. The computational results and discussions are given in section 6.

2. Quartic B-spline Functions

Let us divide the interval $[a,b]$ into $n+1$ equidistant knots $x_i = x_0 + ih, i \in \mathbb{Z}$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, where $n \in \mathbb{Z}^+$ and $h = \frac{1}{n}(b-a)$. We extend $[a,b]$ to $[a-4h, b+4h]$ with equidistant knots $x_i = a + ih$, $i = -4, -3, -2, \cdots, n+4$, and define the typical QBS functions as [9, 19]

$$B_i(x) = \frac{1}{24h^4}\begin{cases} (x-x_{i-2})^4, & x_{i-2} \le x \le x_{i-1} \\ h^4 + 4h^3(x-x_{i-1}) + 6h^2(x-x_{i-1})^2 & \\ +4h(x-x_{i-1})^3 - 4(x-x_{i-1})^4, & x_{i-1} \le x \le x_i \\ 11h^4 + 12h^3(x-x_i) - 6h^2(x-x_i)^2 & \\ -12h(x-x_i)^3 + 6(x-x_i)^4, & x_i \le x \le x_{i+1} \\ h^4 - 4h^3(x-x_{i+2}) + 6h^2(x-x_{i+2})^2 & \\ -4h(x-x_{i+2})^3 - 4(x-x_{i+2})^4, & x_{i+1} \le x \le x_{i+2} \\ (x-x_{i+3})^4, & x_{i+2} \le x \le x_{i+3} \\ 0 & \text{otherwise.} \end{cases} \tag{2. 3}$$

For a smooth function $u(x)$, there always exists a unique QBS, $U(x)$, which satisfies the given interpolating conditions, s.t.

$$U(x) = \sum_{i=-2}^{n+1} c_i B_i(x), \tag{2. 4}$$

where, the constants, c_i's are to be calculated. Let U_j, m_j, M_j and T_j be the QBS approximations for the unknown function $u(x)$ and its first three derivatives at the j^{th} knot respectively.
Using (2. 3) and (2. 4), we have

$$U_j = U(x_j) = \frac{1}{24}\big(c_{j-2} + 11c_{j-1} + 11c_j + c_{j+1}\big), \tag{2. 5}$$

$$m_j = U'(x_j) = \frac{1}{6h}\big(-c_{j-2} - 3c_{j-1} + 3c_j + c_{j+1}\big), \tag{2. 6}$$

$$M_j = U''(x_j) = \frac{1}{2h^2}\big(c_{j-2} - c_{j-1} - c_j + c_{j+1}\big), \tag{2. 7}$$

$$T_j = U'''(x_j) = \frac{1}{h^3}\big(-c_{j-2} + 3c_{j-1} - 3c_j + c_{j+1}\big). \tag{2. 8}$$

Using (2. 5)–(2. 8), we can establish the following relations [19]

$$m_j = u'(x_j) + \frac{h^4}{720}u^{(5)}(x_j) + \cdots, \tag{2. 9}$$

$$M_j = u''(x_j) - \frac{h^4}{240}u^{(6)}(x_j) + \cdots, \tag{2. 10}$$

$$T_j = u'''(x_j) - \frac{h^2}{12}u^{(5)}(x_j) + \frac{h^4}{240}u^{(7)}(x_j) + \cdots. \tag{2. 11}$$

From (2. 9)–(2. 11), we have

$$\begin{aligned}
\|m_j - u'(x_j)\|_\infty &= \max_{0\le j\le n} \|m_j - u'(x_j)\| = O(h^4),\\
\|M_j - u''(x_j)\|_\infty &= O(h^4),\\
\|T_j - u'''(x_j)\|_\infty &= O(h^2).
\end{aligned}$$

The truncation error in T_j is $O(h^2)$, which gives an adequate reason to establish a new approximation to $u'''(x)$.

3. The New Approximation to $u'''(x)$

We use (2. 10) to expand T_{j-2} at the j^{th} knot, ($j = 2, 3, 4, \cdots, n-2$), as

$$\begin{aligned}
T_{j-2} &= u^{(3)}(x_{j-2}) - \frac{h^2}{12}u^{(5)}(x_{j-2}) + \frac{h^4}{240}u^{(7)}(x_{j-2}) + \cdots\\
&= u^{(3)}(x_j) - 2hu^{(4)}(x_j) + \frac{23h^2}{12}u^{(5)}(x_j) - \frac{7h^3}{6}u^{(6)}(x_j) + \frac{121h^4}{240}u^{(7)}(x_j) + \cdots .
\end{aligned}$$

Similar expressions can be derived for T_{j-1}, T_{j+1} and T_{j+2} at j^{th} knot as

$$\begin{aligned}
T_{j-1} &= u^{(3)}(x_j) - hu^{(4)}(x_j) + \frac{5h^2}{12}u^{(5)}(x_j) - \frac{h^3}{12}u^{(6)}(x_j) + \frac{1h^4}{240}u^{(7)}(x_j) + \cdots ,\\
T_{j+1} &= u^{(3)}(x_j) + hu^{(4)}(x_j) + \frac{5h^2}{12}u^{(5)}(x_j) + \frac{h^3}{12}u^{(6)}(x_j) + \frac{1h^4}{240}u^{(7)}(x_j) + \cdots ,\\
T_{j+2} &= u^{(3)}(x_j) + 2hu^{(4)}(x_j) + \frac{23h^2}{12}u^{(5)}(x_j) + \frac{7h^3}{6}u^{(6)}(x_j) + \frac{121h^4}{240}u^{(7)}(x_j) + \cdots .
\end{aligned}$$

We suppose that the new approximation $\widetilde{T}_j$ for $u'''(x_j)$ is given by

$$\widetilde{T}_j = A_1T_{j-2} + A_2T_{j-1} + A_3T_j + A_4T_{j+1} + A_5T_{j+2}. \tag{3. 12}$$

This expression (3. 12) leads to the following five equations involving A_i's as

$$\begin{aligned}
A_1 + A_2 + A_3 + A_4 + A_5 &= 1,\\
-2A_2 - A_3 + A_4 + 2A_5 &= 0,\\
23A_1 + 5A_2 - A_3 + 5A_4 + 23A_5 &= 0,\\
-14A_1 - A_2 + A_4 + 14A_5 &= 0,\\
121A_1 + A_2 + A_3 + A_4 + 121A_5 &= 0.
\end{aligned}$$

Hence, $A_1 = -\frac{1}{240},\ A_2 = \frac{1}{10},\ A_3 = \frac{97}{120},\ A_4 = \frac{1}{10}$, and $A_5 = -\frac{1}{240}$. Substituting A_i's into (3. 12), we get

$$\widetilde{T}_j = \frac{1}{240h^3}\big(c_{j-4} - 27c_{j-3} - 119c_{j-2} + 485c_{j-1} - 485c_j + 119c_{j+1} + 27c_{j+2} - c_{j+3}\big). \tag{3. 13}$$

Now we approximate $u'''(x)$ at the knot x_0 using four neighbouring values, such that

$$\widetilde{T}_0 = A_1T_0 + A_2T_1 + A_3T_2 + A_4T_3, \tag{3. 14}$$

where

$$\begin{aligned}
T_0 &= u^{(3)}(x_0) - \frac{h^2}{12}u^{(5)}(x_0) + \frac{h^4}{240}u^{(7)}(x_0) + \cdots ,\\
T_1 &= u^{(3)}(x_0) + hu^{(4)}(x_0) + \frac{5h^2}{12}u^{(5)}(x_0) + \frac{h^3}{12}u^{(6)}(x_0) + \frac{h^4}{240}u^{(7)}(x_0) + \cdots ,\\
T_2 &= u^{(3)}(x_0) + 2hu^{(4)}(x_0) + \frac{23h^2}{12}u^{(5)}(x_0) + \frac{7h^3}{6}u^{(6)}(x_0) + \frac{121h^4}{240}u^{(7)}(x_0) + \cdots ,\\
T_3 &= u^{(3)}(x_0) + 3hu^{(4)}(x_0) + \frac{53h^2}{12}u^{(5)}(x_0) + \frac{17h^3}{4}u^{(6)}(x_0) + \frac{721h^4}{240}u^{(7)}(x_0) + \cdots .
\end{aligned}$$

The expression (3. 14) yields the following four equations

$$\begin{aligned}
A_1 + A_2 + A_3 + A_4 &= 1,\\
A_2 + 2A_3 + 3A_4 &= 0,\\
-A_1 + 5A_2 + 23A_3 + 53A_4 &= 0,\\
A_2 + 14A_3 + 51A_4 &= 0.
\end{aligned}$$

Hence, $A_1 = \frac{7}{6},\ A_2 = -\frac{5}{12},\ A_3 = \frac{1}{3}$ and $A_4 = -\frac{1}{12}$. Using these values in (3. 14), we obtain

$$\widetilde{T}_0 = \frac{1}{12h^3}\big(- 14c_{-2} + 47c_{-1} - 61c_0 + 42c_1 - 20c_2 + 7c_3 - c_4\big). \tag{3. 15}$$

Now, using four neighbouring knots at x_1, we suppose

$$\widetilde{T}_1 = A_1T_0 + A_2T_1 + A_3T_2 + A_4T_3, \tag{3. 16}$$

where

$$\begin{aligned}
T_0 &= u^{(3)}(x_1) - hu^{(4)}(x_1) + \frac{5h^2}{12}u^{(5)}(x_1) - \frac{h^3}{12}u^{(6)}(x_1) + \frac{h^4}{240}u^{(7)}(x_1) + \cdots ,\\
T_1 &= u^{(3)}(x_1) - \frac{h^2}{12}u^{(5)}(x_1) + \frac{h^4}{240}u^{(7)}(x_1) + \cdots ,\\
T_2 &= u^{(3)}(x_1) + hu^{(4)}(x_1) + \frac{5h^2}{12}u^{(5)}(x_1) + \frac{h^3}{12}u^{(6)}(x_1) + \frac{h^4}{240}u^{(7)}(x_1) + \cdots ,\\
T_3 &= u^{(3)}(x_1) + 2hu^{(4)}(x_1) + \frac{23h^2}{12}u^{(5)}(x_1) + \frac{7h^3}{6}u^{(6)}(x_1) + \frac{121h^4}{240}u^{(7)}(x_1) + \cdots .
\end{aligned}$$

Solving the above system, we get $A_1 = \frac{1}{12},\ A_2 = \frac{5}{6},\ A_3 = \frac{1}{12}$ and $A_4 = 0$. Substituting A_i's back into (3. 16), we have

$$\widetilde{T}_1 = \frac{1}{12h^3}\big(- c_{-2} - 7c_{-1} + 26c_0 - 26c_1 + 7c_2 + c_3\big). \tag{3. 17}$$

In a very similar fashion, involving four neighbouring knots, we can establish the following approximations at x_{n-1} and x_n respectively

$$\widetilde{T}_{n-1} =\frac{1}{12h^3}\big(- c_{n-4} - 7c_{n-3} + 26c_{n-2} - 26c_{n-1} + 7c_n + c_{n+1}\big), \tag{3. 18}$$

$$\widetilde{T}_n =\frac{1}{12h^3}\big(c_{n-5} - 7c_{n-4} + 20c_{n-3} - 42c_{n-2} + 61c_{n-1} - 47c_n + 14c_{n+1}\big). \tag{3. 19}$$

4. Description of the Numerical Method

Employing Quasi-linearization technique, equation (1. 1) is transformed as

$$\alpha u'''_{m+1}(x)+\frac{\beta}{x}u''_{m+1}(x)+r(x)u'_{m+1}(x)+Y_m(x)u_{m+1}(x)=Z_m(x),\ \ x\in[0,1], \quad (4.\ 20)$$

where $Y_m(x)=-\big(\frac{\partial f}{\partial u}\big)_{(x,u_m)}$ and $Z_m(x)=f(x,u_m)-\big(\frac{\partial f}{\partial u}\big)_{(x,u_m)}$, $m=0,1,2,\cdots$. The end conditions (1. 2) are also reshaped as

$$u_{m+1}(0)=\alpha_1,\ u'_{m+1}(0)=\alpha_2,\ u_{m+1}(1)=\alpha_3. \quad (4.\ 21)$$

Using L' Hopital's rule, equation (4. 20) is modified as

$$p(x)u'''_{m+1}(x)+q(x)u''_{m+1}(x)+r(x)u'_{m+1}(x)+Y_m(x)u_{m+1}(x)=Z_m(x),\ \ x\in[0,1], \quad (4.\ 22)$$

where $p(x)=\begin{cases}\alpha+\beta, & \text{if } x=0\\ \alpha, & \text{if } x\neq 0\end{cases}$ and $q(x)=\begin{cases}0, & \text{if } x=0\\ \frac{\beta}{x}, & \text{if } x\neq 0.\end{cases}$

We suppose that the quartic B-spline solution to (4. 22) is given by

$$U(x)=\sum_{i=-2}^{n+1}c_iB_i(x). \quad (4.\ 23)$$

Discretizing (4. 22) at the j^{th} knot, we obtain

$$p(x_j)U'''_{m+1}(x_j)+q(x_j)U''_{m+1}(x_j)+r(x_j)U'_{m+1}(x_j)+Y_m(x_j)U_{m+1}(x_j)=Z_m(x_j). \quad (4.\ 24)$$

For $j=2,3,4,\cdots,n-2$, we use (2. 5)–(2. 7) and (3. 13) in (4. 24) to obtain the following equations

$$\begin{aligned}p(x_j)&\left(\frac{c_{j-4}-27c_{j-3}-119c_{j-2}+485c_{j-1}-485c_j+119c_{j+1}+27c_{j+2}-c_{j+3}}{240h^3}\right)\\ &+q(x_j)\left(\frac{c_{j-2}-c_{j-1}-c_j+c_{j+1}}{2h^2}\right)+r(x_j)\left(\frac{-c_{j-2}-3c_{j-1}+3c_j+c_{j+1}}{6h}\right)\\ &+Y_m(x_j)\big(\frac{c_{j-2}+11c_{j-1}+11c_j+c_{j+1}}{24}\big)=Z_m(x_j).\end{aligned} \quad (4.\ 25)$$

Similarly, at the knots $x_0,\ x_1,\ x_{n-1}$ and x_n, (4. 24) yields the following equations respectively

$$\begin{aligned}p(x_0)&\left(\frac{-14c_{-2}+47c_{-1}-61c_0+42c_1-20c_2+7c_3-c_4}{12h^3}\right)\\ &+q(x_0)\left(\frac{c_{-2}-c_{-1}-c_0+c_1}{2h^2}\right)+r(x_0)\left(\frac{-c_{-2}-3c_{-1}+3c_0+c_1}{6h}\right)\\ &+Y_m(x_0)\big(\frac{c_{-2}+11c_{-1}+11c_0+c_1}{24}\big)=Z_m(x_0),\end{aligned} \quad (4.\ 26)$$

$$p(x_1)\Bigg(\frac{-c_{-2}-7c_{-1}+26c_0-26c_1+7c_2+c_3}{12h^3}\Bigg) + q(x_1)\Bigg(\frac{c_{-1}-c_0-c_1+c_2}{2h^2}\Bigg)+r(x_1)\Bigg(\frac{-c_{-1}-3c_0+3c_1+c_2}{6h}\Bigg) + Y_m(x_1)\big(\frac{c_{-1}+11c_0+11c_1+c_2}{24}\big)=Z_m(x_1), \quad (4.\ 27)$$

$$p(x_{n-1})\Bigg(\frac{-c_{n-4}-7c_{n-3}+26c_{n-2}-26c_{n-1}+7c_n+c_{n+1}}{12h^3}\Bigg) + q(x_{n-1})\Bigg(\frac{c_{n-3}-c_{n-2}-c_{n-1}+c_n}{2h^2}\Bigg)+r(x_{n-1})\Bigg(\frac{-c_{n-3}-3c_{n-2}+3c_{n-1}+c_n}{6h}\Bigg) + Y_m(x_{n-1})\big(\frac{c_{n-3}+11c_{n-2}+11c_{n-1}+c_n}{24}\big)=Z_m(x_{n-1}), \quad (4.\ 28)$$

$$p(x_n)\Bigg(\frac{c_{n-5}-7c_{n-4}+20c_{n-3}-42c_{n-2}+61c_{n-1}-47c_n+14c_{n+1}}{12h^3}\Bigg) + q(x_n)\Bigg(\frac{c_{n-2}-c_{n-1}-c_n+c_{n+1}}{2h^2}\Bigg)+r(x_n)\Bigg(\frac{-c_{n-2}-3c_{n-1}+3c_n+c_{n+1}}{6h}\Bigg) + Y_m(x_n)\big(\frac{c_{n-2}+11c_{n-1}+11c_n+c_{n+1}}{24}\big)=Z_m(x_n). \quad (4.\ 29)$$

The set of boundary conditions (4. 21) as well give the following three equations

$$\frac{c_{-2}+11c_{-1}+11c_0+c_1}{24} = \alpha_1, \quad (4.\ 30)$$

$$\frac{-c_{-2}-3c_{-1}+3c_0+c_1}{6h} = \alpha_2, \quad (4.\ 31)$$

$$\frac{c_{n-2}+11c_{n-1}+11c_n+c_{n+1}}{24} = \alpha_3. \quad (4.\ 32)$$

The set of equations (4. 25)–(4. 32), with unknowns c_i's, $i=-2,-1,0,\cdots,n+1$, can be expressed in matrix notation as

$$Ac-b=0 \quad (4.\ 33)$$

The matrix equation (4. 33) represents a system of $n+4$ linear equations involving $n+4$ unknowns. Setting $m=0$, we begin with an initial guess for $U_0(x)$ and solve (4. 33) for c using a modified form of well known Thomas algorithm [9, 19, 20]. The values of c_i's are then put into (4. 23) to get $U_{m+1}(x)$. This procedure is continued for $m=1,2,3,\cdots$ till $max|U_{m+1}(x)-U_m(x)|\leq 10^{-8}$. The numerical calculations are carried out using $Mathematica$ 9.

5. Error Analysis

Using (2. 5)–(2. 7), we can establish the following relations [9]

$$h\big[U'(x_{j-2})+11U'(x_{j-1})+11U'(x_j)+U'(x_{j+1})\big]=4\big[U(x_{j+1})+3U(x_j) - 3U(x_{j-1})-U(x_{j-2})\big], \quad (5.\ 34)$$

$$h^2U''(x_j) = 2\big[U(x_{j+1}) - 2U(x_j) + U(x_{j-1})\big] - \frac{h}{2}\big[U'(x_{j+1}) - U'(x_{j-1})\big]. \quad (5.\ 35)$$

Similarly, using (3. 13), we have

$$\begin{aligned} h^3U'''(x_j) = \frac{h}{20}\big[59U'(x_{j-1}) - 72U'(x_j) + 13U(x_{j+1})\big] + \frac{h^2}{120}\big[U''(x_{j-2}) \\ + 92U''(x_{j-1}) + 184U''(x_j) - U''(x_{j+2})\big]. \quad (5.\ 36) \end{aligned}$$

Employing the operator notation, $E^\mu(U'(x_j)) = U'(x_{j+\mu})$, $\mu \in \mathbb{Z}$, the relation (5. 34) can be expressed as

$$h[E^{-2} + 11E^{-1} + 11E^0 + E^1]U'(x_j) = 4[E^1 + 3E^0 - 3E^{-1} - E^{-2}]u(x_j).$$

Hence,

$$hU'(x_j) = 4\left[\frac{E + 3 - 3E^{-1} - E^{-2}}{E^{-2} + 11E^{-1} + 11 + E}\right]u(x_j). \quad (5.\ 37)$$

Using $D \equiv d/dx$ and $E = \mathrm{e}^{hD}$, we get

$$\begin{aligned} E + 3 - 3E^{-1} - E^{-2} &= \mathrm{e}^{hD} + 3 - 3\mathrm{e}^{-hD} - \mathrm{e}^{-2hD} \\ &= 24hD - 12h^2D^2 + 8h^3D^3 - 3h^4D^4 + \cdots, \end{aligned}$$

$$\begin{aligned} E^{-2} + 11E^{-1} + 11 + E &= \mathrm{e}^{-2hD} + 11\mathrm{e}^{-hD} + 11 + \mathrm{e}^{hD} \\ &= 24 - 12hD + 8h^2D^2 - 3h^3D^3 + \frac{7}{6}h^4D^4 + \cdots. \end{aligned}$$

Therefore, equation (5. 37) can be expressed as

$$\begin{aligned} U'(x_j) &= \left(D - \frac{1}{2}hD^2 + \frac{1}{3}h^2D^3 + \cdots\right)\left(1 - \frac{1}{2}hD + \frac{1}{3}h^2D^2 + \cdots\right)^{-1}u(x_j) \\ &= \left(D - \frac{1}{2}hD^2 + \frac{1}{3}h^2D^3 + \cdots\right)\left[1 + \left(-\frac{1}{2}hD + \frac{1}{3}h^2D^2 + \cdots\right)\right]^{-1}u(x_j) \\ &= \left(D - \frac{1}{2}hD^2 + \frac{1}{3}h^2D^3 + \cdots\right)\left[1 + \frac{1}{2}hD - \frac{1}{12}h^2D^2 + \cdots\right]u(x_j) \\ &= \left(D + \frac{1}{720}h^4D^5 - \frac{1}{2016}h^6D^7 + \cdots\right)u(x_j). \end{aligned}$$

Simplifying, we obtain

$$U'(x_j) = u'(x_j) + \frac{1}{720}h^4u^{(5)}(x_j) - \frac{1}{2016}h^6u^{(7)}(x_j) + \cdots. \quad (5.\ 38)$$

Similarly, (5. 35) can be expressed as

$$\begin{aligned} U''(x_j) &= \frac{2}{h^2}\left[E^{-1} - 2E^0 + E^1\right]u(x_j) - \frac{1}{2h}\big[E^1 - E^{-1}\big]u'(x_j) \\ &= \left(2h^2D^2 + \frac{h^4D^4}{6} + \frac{h^6D^6}{180} + \frac{h^8D^8}{10080} + \cdots\right)u(x_j) \\ &\quad - \left(h^2D + \frac{h^4D^3}{6} + \frac{h^6D^5}{120} + \frac{h^8D^7}{5040} + \cdots\right)u'(x_j). \end{aligned}$$

After simplification, the above relation yields the following expression

$$U''(x_j) = u''(x_j) - \frac{h^4}{240}u^{(6)}(x_j) + \frac{h^6}{6048}u^{(8)}(x_j) + \cdots . \quad (5.\ 39)$$

In the same way, (5. 36) gives the following relation

$$U'''(x_j) = u'''(x_j) + \frac{23h^3}{1800}u^{(6)}(x_j) - \frac{7h^4}{1200}u^{(7)}(x_j) + \frac{23h^5}{37800}u^{(8)}(x_j)\cdots . \quad (5.\ 40)$$

We define the error term at j^{th} knot as $e(x_j) = U(x_j) - u(x_j)$. Using (5. 38)-(5. 40) in Taylor series of error term, we obtain

$$e(x_j + \phi h) = \frac{\phi}{720}h^5u^{(5)}(x_j) + \frac{\phi^2(-45+46\phi)}{21600}h^6u^{(6)}(x_j) + \cdots , \quad (5.\ 41)$$

where, $0 \leq \phi \leq 1$. From (5. 41), it is clear that the truncation error in the new QBS approximation method is $O(h^5)$.

6. Numerical Results

In this section, we discuss the approximate solution of (1. 1) by new QBS approximation technique. In order to test the accuracy of proposed numerical technique, the error norm L_∞ is evaluated as

$$L_\infty = \|U_j - u_j\|_\infty = \max_{0\leq j\leq n}|U(x_j) - u(x_j)|.$$

Example 6.1. *Consider the following SBVP* [4, 11]

$$u'''(x) - \frac{2}{x}u''(x) - u(x) - u^2(x) = -6\mathrm{e}^x + 6x\mathrm{e}^x + 7x^2\mathrm{e}^x - x^6\mathrm{e}^{2x},\ \ x \in [0,1],$$

with the end conditions

$$u(0) = 0,\ \ u'(0) = 0,\ \ u(1) = \mathrm{e}.$$

The closed form solution is $x^3\mathrm{e}^x$. The computational results are listed in Table 1. It is found that our approximate results are better than DTM [4], MADM [11] and quartic B-spline collocation method (QBSM) used in [20]. Figure 1 displays the analytical and numerical solution when $h = 1/20$. The absolute computational error corresponding to four different values of h is shown in Figure 2.

Example 6.2. *Consider the* 3^{rd} *order SBVP* [4, 12, 13]

$$u'''(x) - \frac{2}{x}u''(x) - u^3(x) = -6\mathrm{e}^x + 6x\mathrm{e}^x + 7x^2\mathrm{e}^x + x^3\mathrm{e}^x - x^9\mathrm{e}^{3x},\ \ x \in [0,1],$$

with following end conditions

$$u(0) = 0,\ \ u'(0) = 0,\ \ u'(1) = 4\mathrm{e}.$$

The closed form solution is $u(x) = x^3\mathrm{e}^x$. In Table 2–3, the approximate results are compared with DTM [4], MADM [12], NCBSM [13] and QBSM used in [20]. From Figure 3, one can clearly observe that the computational error decreases with h.

TABLE 1. Computational results for Example 6.1.

x	DTM [4]	MADM [11]	QBSM $h=1/40$	Proposed method $h=1/20$	Proposed method $h=1/40$	Exact solution
0.0	0	0	0.0000000000	0.0000000000	0.0000000000	0
0.1	0.0011051737	0.0011049829	0.0011044865	0.0011051345	0.0011051687	0.0011051709
0.2	0.0097712661	0.0097682140	0.0097659876	0.0097711892	0.0097712202	0.0097712221
0.3	0.0364464109	0.0364309587	0.0364291591	0.0364462015	0.0364461889	0.0364461878
0.4	0.0954774849	0.0954286436	0.0954390698	0.0954768795	0.0954767871	0.0954767806
0.5	$\cdots$	$\cdots$	0.2060239268	0.2060903684	0.2060901723	0.2060901588
0.6	0.3935811403	0.3933337725	0.3934796306	0.3935779841	0.3935776815	0.3935776609
0.7	0.6907232803	0.6902648178	0.6905929245	0.6907175863	0.6907172045	0.6907171787
0.8	1.1394859561	1.1387033873	1.1393459652	1.1394773753	1.1394769819	1.1394769554
0.9	1.7930602392	1.7918056879	1.7929522212	1.7930509742	1.7930506873	1.7930506680
1.0	2.7182818285	2.7163675037	2.7182818285	2.7182818285	2.7182818285	2.7182818285
L_∞	9.57×10^{-6}	1.91×10^{-3}	1.32×10^{-4}	4.26×10^{-7}	2.70×10^{-8}	$\cdots$

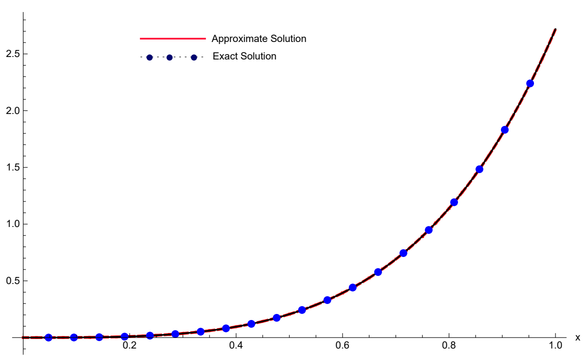
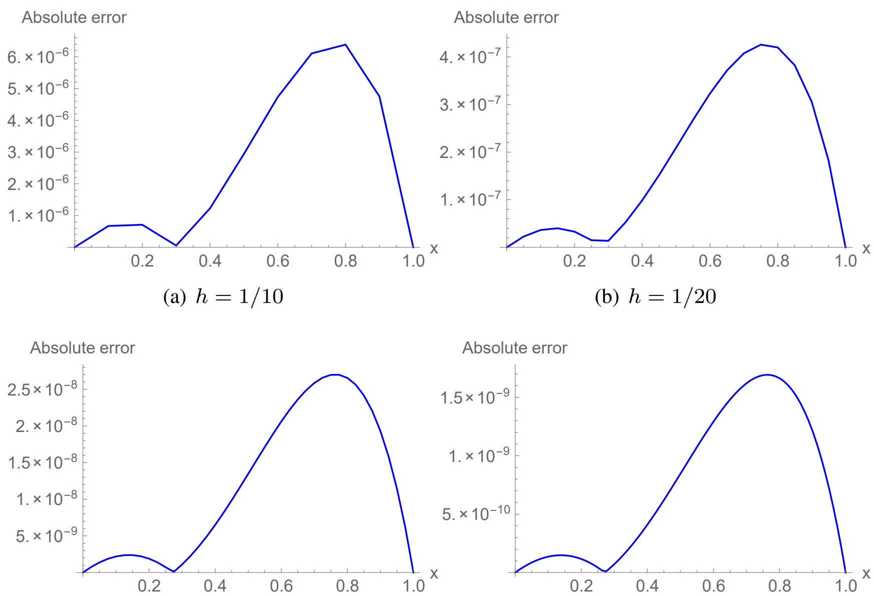


FIGURE 1. Numerical and analytical solution for Example 6.1 when $h = 1/20$.

TABLE 2. Computational results for Example 6.2.

x	DTM [4]	MADM [12]	QBSM $h=1/40$	Proposed method $h=1/20$	Proposed method $h=1/40$	Exact solution
0.0	0	0	0.0000000000	0.0000000000	0.0000000000	0
0.1	0.0011049829	0.0011027286	0.0011044856	0.0011051345	0.0011051687	0.0011051709
0.2	0.0097682140	0.0097321453	0.0097659741	0.0097711896	0.0097712202	0.0097712221
0.3	0.0364309587	0.0362483607	0.0364290953	0.0364462036	0.0364461890	0.0364461878
0.4	0.0954286436	0.0948515377	0.0954388885	0.0954768863	0.0954767876	0.0954767806
0.5	0.2059726070	0.2045635871	0.2060235481	0.2060903850	0.2060901734	0.2060901588
0.6	0.3933337725	0.3904115919	0.3934790076	0.3935780184	0.3935776837	0.3935776609
0.7	0.6902648178	0.6848491728	0.6905921151	0.6907176489	0.6907172086	0.6907171787
0.8	1.1387033873	1.1294576080	1.1393451901	1.1394774804	1.1394769889	1.1394769554
0.9	1.7918056879	1.7769747466	1.7929517917	1.7930511407	1.7930506983	1.7930506680
1.0	2.7163675037	2.6937066308	2.7182818285	2.7182820822	2.7182818452	2.7182818285
L_∞	1.90×10^{-3}	2.46×10^{-2}	1.33×10^{-4}	5.25×10^{-7}	3.35×10^{-8}	$\cdots$



(a) $h = 1/10$ (b) $h = 1/20$

(c) $h = 1/40$ (d) $h = 1/80$

FIGURE 2. Computational error norm for Example 6.1.

TABLE 3. Absolute numerical error for Example 6.2.

n	QBSM	NCBSM [13]	Proposed method
30	2.37×10^{-4}	3.98×10^{-5}	1.05×10^{-7}
40	1.33×10^{-4}	8.96×10^{-6}	3.35×10^{-8}

Example 6.3. *Consider the* 3^{rd} *order SBVP* [22]

$$u'''(x) + \frac{3}{x}u''(x) - u^3(x) = 24\mathrm{e}^x + 36x\mathrm{e}^x + 12x^2\mathrm{e}^x + x^3\mathrm{e}^x - x^9\mathrm{e}^{3x}, \ \ x \in [0,1],$$

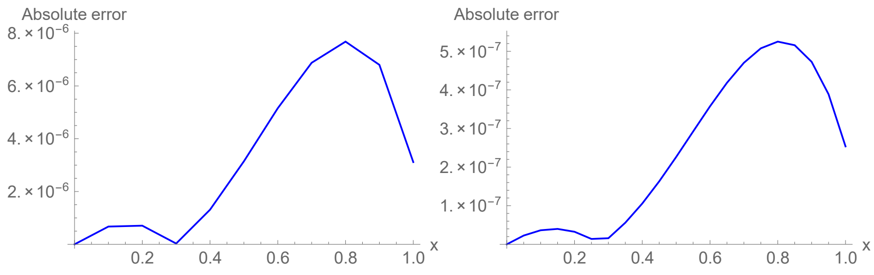
with the end conditions

$$u(0) = 0, \ \ u'(0) = 0, \ \ u(1) = \mathrm{e}.$$

The analytical exact solution is $u(x) = x^3\mathrm{e}^x$. The computational results are tabulated in Table 4. It can be concluded that our results are better than QBSM used in [20] and Iterative decomposition method (IDM) [22]. The absolute numerical error for different choices of mesh size is displayed in Figure 4.

Example 6.4. *Consider the non–linear singular initial value problem* [7, 24]

$$u'''(x) + \frac{2}{x}u''(x) = \frac{9(x^6+8)}{8u^5(x)}, \ \ x \in [0,1],$$



(a) $h = 1/10$ (b) $h = 1/20$

(c) $h = 1/40$ (d) $h = 1/80$

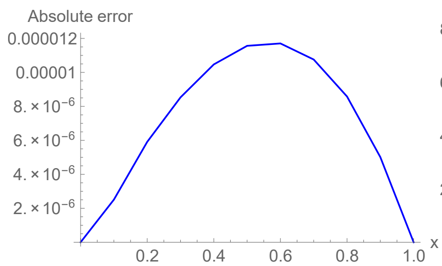
FIGURE 3. Computational error norm for Example 6.2.

TABLE 4. Numerical results for Example 6.3.

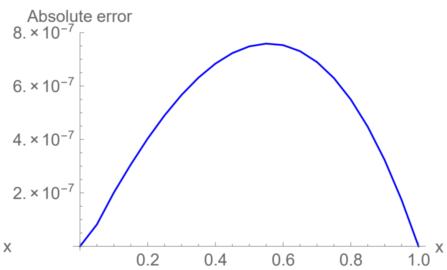
x	IDM [22]	QBSM $h = 1/40$	Proposed method $h = 1/20$	Proposed method $h = 1/40$	Exact solution
0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	0.0011050170	0.0010769248	0.0011049698	0.0011051573	0.0011051709
0.2	0.0097699780	0.0097141285	0.0097708186	0.0097711958	0.0097712221
0.3	0.0364419458	0.0363625885	0.0364456211	0.0364461515	0.0364461878
0.4	0.0954666206	0.0953705090	0.0954760967	0.0954767371	0.0954767806
0.5	0.2060700980	0.2059671104	0.2060894103	0.2060901114	0.2060901588
0.6	0.3935425575	0.3934463760	0.3935769077	0.3935776133	0.3935776609
0.7	0.6906604686	0.6905895196	0.6907164889	0.6907171351	0.6907171787
0.8	1.1393898620	1.1393688849	1.1394764059	1.1394769207	1.1394769554
0.9	1.7929200380	1.7929831323	1.7930503453	1.7930506477	1.7930506680
1.0	2.7170847700	2.7182818285	2.7182818285	2.7182818285	2.7182818285
L_∞	1.20×10^{-3}	1.32×10^{-4}	7.59×10^{-7}	4.80×10^{-8}	$\cdots$

with the initial conditions

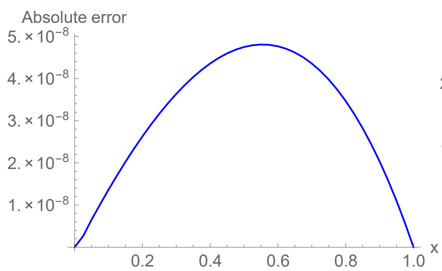
$$u(0) = 1, \;\; u'(0) = u''(0) = 0.$$



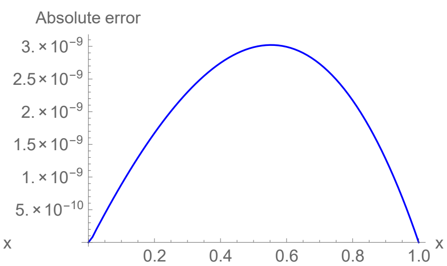
(a) $h = 1/10$



(b) $h = 1/20$



(c) $h = 1/40$



(d) $h = 1/80$

FIGURE 4. Absolute computational error for Example 6.3.

The exact solution is $\sqrt{1+x^3}$. The comparison of computational outcomes with RKM [7], QBSM used in [20], ADM [23] and VIM [24] is presented in Table 5–6. It is observed that our proposed approximation technique shows superior results as compared to QBSM and VIM. The exact and numerical solutions are shown in Figure 5. The computational error norm with $n = 10, 20, 40, 80$ has been portrayed in Figure 6.

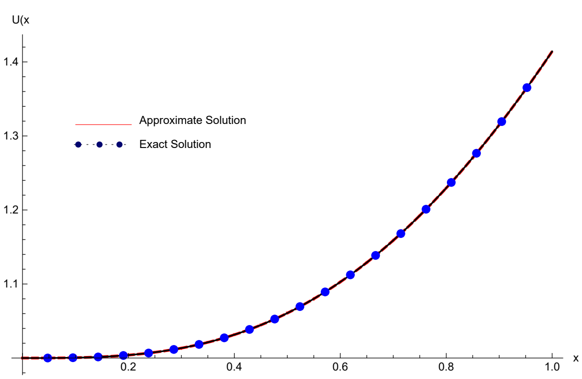


FIGURE 5. Exact and approximate solution for Example 6.4 when $h = 1/20$.

TABLE 5. Approximate results for Example 6.4.

x	ADM [23] VIM [24]	QBSM $h = 1/40$	Proposed method $h = 1/20$	Proposed method $h = 1/40$	Exact solution
0.0	1	1.0000000000	1.0000000000	1.0000000000	1
0.1	1.0004998751	1.0004998646	1.0004998639	1.0004998743	1.0004998751
0.2	1.0039920318	1.0039918750	1.0039919869	1.0039920290	1.0039920318
0.3	1.0134100844	1.0134093262	1.0134099869	1.0134100786	1.0134100848
0.4	1.0315037286	1.0315015223	1.0315035946	1.0315037464	1.0315037567
0.5	1.0606594086	1.0606552770	1.0606599467	1.0606601576	1.0606601718
0.6	1.1027128254	1.1027151874	1.1027236267	1.1027238830	1.1027239002
0.7	1.1587753002	1.1588655553	1.1588784721	1.1588787499	1.1588787685
0.8	1.2289362534	1.2296165165	1.2296338035	1.2296340739	1.2296340919
0.9	1.3112512518	1.3148935619	1.3149141957	1.3149144303	1.3149144459
1.0	1.3984375	1.4141911866	1.4142133767	1.4142135508	1.4142135624
L_∞	1.58×10^{-2}	2.24×10^{-5}	2.96×10^{-7}	1.86×10^{-8}	$\cdots$

TABLE 6. Absolute computational error for Example 6.4.

n	QBSM	RKM [7]	Proposed method
20	8.97×10^{-5}	7.78×10^{-6}	2.96×10^{-7}
35	2.92×10^{-5}	4.40×10^{-6}	3.18×10^{-8}

TABLE 7. Absolute computational errors for Example 6.5 when $\epsilon = 10^{-2}$.

n	16	32	64	128	256
Proposed method	3.16×10^{-3}	9.77×10^{-4}	5.37×10^{-5}	3.43×10^{-6}	2.17×10^{-7}
QBSM [20]	2.10×10^{-2}	6.59×10^{-3}	1.50×10^{-3}	3.77×10^{-4}	9.55×10^{-5}

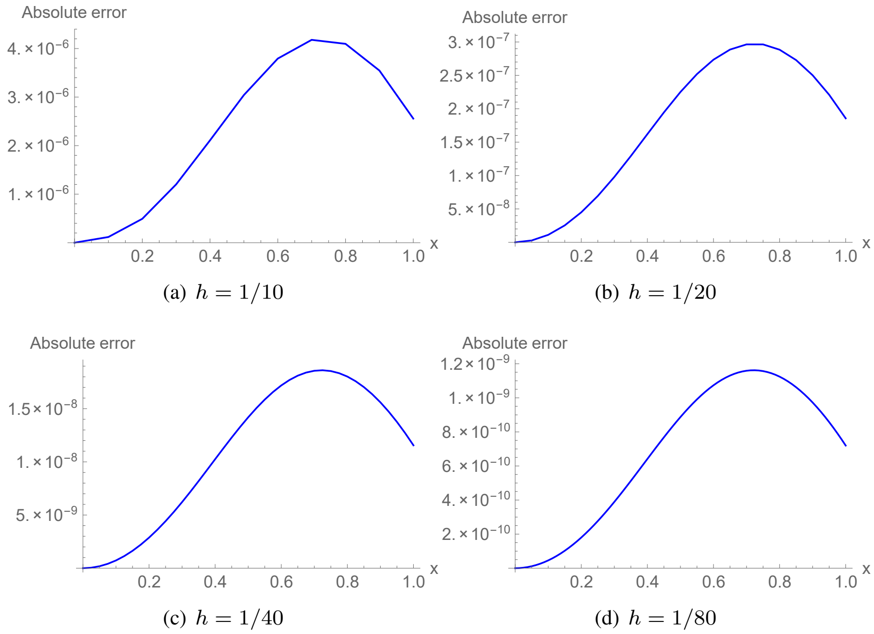
Example 6.5. *Consider the singularly perturbed SBVP* [20]

$$\epsilon u'''(x) + \frac{2}{x}u''(x) + u'(x) + u(x) = \left(1 - \frac{2}{\epsilon x}\right)\frac{\sin\left(\frac{x}{\sqrt{\epsilon}}\right)}{\sin\left(\frac{1}{\sqrt{\epsilon}}\right)}, \;\; x \in [0,1],$$

with the end conditions

$$u(0) = 0, \;\; u(1) = 1, \;\; u'(1) = \frac{\cos\left(\frac{1}{\sqrt{\epsilon}}\right)}{\sqrt{\epsilon}\sin\left(\frac{1}{\sqrt{\epsilon}}\right)}.$$

The analytical exact solution is $\frac{\sin(\frac{x}{\sqrt{\epsilon}})}{\sin(\frac{1}{\sqrt{\epsilon}})}$. The computational error norm for different values of n is presented in Table 7. It is revealed that our computed results are better than QBSM [20]. In Figure 7–8, the approximate and exact solutions are exhibited for two different values of perturbation constant when $n = 20$.



(a) $h = 1/10$ (b) $h = 1/20$

(c) $h = 1/40$ (d) $h = 1/80$

FIGURE 6. Computational error norm for Example 6.4.

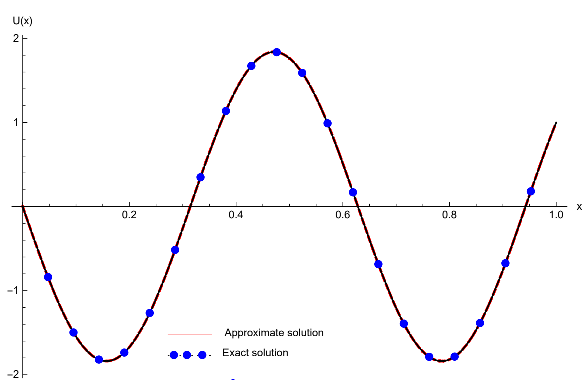


FIGURE 7. Exact and numerical solution for Example 6.5 when $n = 20$, $\epsilon = 10^{-2}$.

7. CONCLUSIONS

In this article, numerical solution of 3^{rd} order SBVP's has been studied. The proposed scheme is applied on several test problems and we conclude the following outcomes

(1) The proposed numerical method is based on a new quartic B-spline approximation for 3^{rd} order derivative.

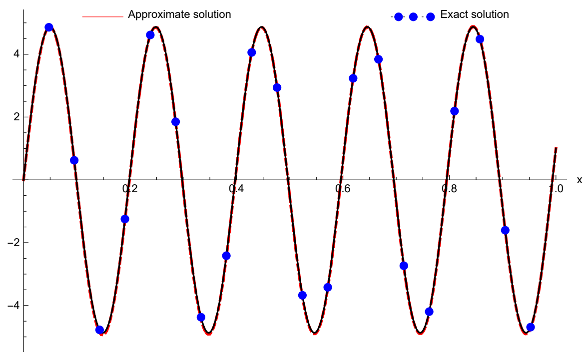


FIGURE 8. Exact and approximate solution for Example 6.5 when $n = 20$, $\epsilon = 10^{-3}$.

(2) The presented approximation is novel for third order SBVP's.
(3) The truncation error in the proposed numerical algorithm is $O(h^5)$.
(4) As the step size is decreased, the computational error decreases.
(5) Due to its straightforward and simple application, our new approximation technique produces more accurate results as compared to DTM [4], RKM [7], MADM [11, 12], NCBSM [13], QBSM used in [20], IDM [22], ADM [23] and VIM [24].

REFERENCES

[1] M. Abukhaled, S. A. Khuri and A. Sayfy, *A numerical approach for solving a class of singular boundary value problems arising in physiology*, IJNAM, **8**, No. 2 (2011) 353-363.
[2] G. Akram, *Quartic spline solution of a third order singularly perturbed boundary value problem*, ANZIAM J. **53**, (2012) 44-58.
[3] M. A. Ali, M. Rafiq and M. O. Ahmad, *Numerical Analysis of a Modified SIR Epidemic Model with the Effect of Time Delay*, Punjab Univ. j. math. **51**, No. 2 (2019) 79-90.
[4] K. Aruna and ASV. R. Kanth, *A novel approach for a class of higher order nonlinear singular boundary value problems*, IJPAM. **84**, No. 4 (2013) 321-329.
[5] H. N. Caglar and S. H. Caglar, *B-spline solution of singular boundary value problems*, Appl. Math. Comput. **182**, No. 2 (2006) 1509-1513.
[6] H. N. Caglar, S. H. Caglar and M. Ozer, *B-spline solution of non-linear singular boundary value problems arising in physiology*, Chaos, Solitons & Fractals, **39**, No. 3 (2009) 1232-1237.
[7] A. Dezhbord, L. Taher and M. Katayoun, *A numerical approach for solving the high-order nonlinear singular EmdenFowler type equations*, Adv. Differ. Equ. **1**, (2018) 161.
[8] J. Goh, A. Abd. Majid and A. I. Md. Ismail, *Extended cubic uniform B-spline for a class of singular boundary value problems*, Nucl. Phys. **2**, (2011).
[9] J. Goh, A. Abd. Majid and A. I. Md. Ismail, *A quartic B-spline for second-order singular boundary value problems*, Comput. & Math. Appl. **64**, No. 2 (2012) 115-120.
[10] F. Haq, K. Shah, A. Khan, M. Shahzad and G. U. Rahman, *Numerical Solution of Fractional Order Epidemic Model of a Vector Born Disease by Laplace Adomian Decomposition Method*, Punjab Univ. j. math. **49**, No. 2 (2017) 13-22.
[11] Y. Q. Hasan and L. M. Zhu, *Solving singular boundary value problems of higher-order ordinary differential equations by modified Adomian decomposition method*, Comm. Nonlinr. Sci. Numr. Simu. **14**, No. 6 (2009) 2592-2596.

[12] Y. Q. Hasan and L. M. Zhu, *A note on the use of modified Adomian decomposition method for solving singular boundary value problems of higher-order ordinary differential equations*, Comm. Nonlinear Sci. Num. Sim. **14**, No. 8 (2009) 3261-3265.

[13] M. K. Iqbal, M. Abbas and I. Wasim, *New cubic B-spline approximation for solving third order Emden-Flower type equations*, Appl. Math. Comput. **331**, (2018) 319-333.

[14] M. K. Iqbal, M. Abbas and N. Khalid, *New Cubic B-spline Approximation for Solving Non-linear Singular Boundary Value Problems Arising in Physiology*, Comm. Math. & Appl. **9**, No. 3 (2018) 377-392.

[15] A. Khalid and M. N. Naeem, *Cubic B-spline Solution of Nonlinear Sixth Order Boundary Value Problems*, Punjab Univ. j. math. **50**, No. 4 (2018) 91-103.

[16] S. A. Khuri, *An Alternative Solution Algorithm for the Nonlinear Generalized Emden-Fowler Equation*, Int. J. Nonlinr. Sci. Numr. Simu. **2**, No. 3 (2001) 299-302.

[17] S. A. Khuri and A. Sayfy, *Numerical solution for the nonlinear Emden–Flower type equations by a fourth-order adaptive method*, Int. J. Comput. Methods, **11**, No. 1 (2014) 21.

[18] W. Kim and C. Chun, *A modified Adomian decomposition method for solving higher-order singular boundary value problems*, Zeitschrift für Naturforschung A. **65**, No. 12 (2010) 1093-1100.

[19] F. G. Lang and XP. Xu, *Quartic B-spline collocation method for fifth order boundary value problems*, Computing, **92**, No. 4 (2011) 365-378.

[20] H. K. Mishra and S. Saini, *Quartic B-spline method for solving singularly perturbed third-order boundary value problems*, Am. J. Numr. Anal. **3**, No. 1 (2015) 18-24.

[21] M. Suleman and S. Riaz, *Unconditionally Stable Numerical Scheme to Study the Dynamics of Hepatitis B Disease*, Punjab Univ. j. math. **49**, No. 3 (2017) 99-118.

[22] O. A. Taiwo and M. O. Hassan, *Approximation of higher-order singular initial and boundary value problems by Iterative decomposition and Bernstein polynomial methods*, British J. Math. Comput. Sci. **9**, No. 6 (2015) 498-515.

[23] A. M. Wazwaz, R. Rach, L. Bougoffa and J. S. Duan, *Solving the Lane-Emden-Fowler type equations of higher orders by the Adomian decomposition method*, Comput. Model. Eng. Sci. **100**, No. 6 (2014) 507-529.

[24] A. M. Wazwaz, *Solving two Emden–Fowler type equations of third order by the Variational iteration method*, Appl. Math. **9**, No. 5 (2015) 2429-2436.

[25] L. J. Xie, C. L. Zhou and S. Xu, *An effective numerical method to solve a class of nonlinear singular boundary value problems using improved differential transform method*, SpringerPlus **5**, No. 1 (2016) 1066.

[26] A. A Zafar, M. B. Riaz, M. A. Imran, *Unsteady Rotational Flow of Fractional Maxwell Fluid in a Cylinder Subject to Shear Stress on the Boundary*, Punjab Univ. j. math. **50**, No. 2 (2018) 21-32.