

**An Application of Product Summability to Approximating the Conjugate Series of a Special Class of Signals**

Rama Chandra Padhy<sup>1,3</sup>, Birupakhya Padhy<sup>2</sup>, Padmanava Samanta<sup>3</sup>, Mahendra Misra<sup>4</sup>,  
Umakanta Misra<sup>5</sup>

<sup>1</sup> Department of Mathematics, Saraswati Degree Vidya Mandir,  
Neelakantha Nagar, Berhampur-760 002, India.  
Email: ramachandrapadhy@gmail.com

<sup>2</sup> Department of Mathematics, KIIT Univeristy, Bhubaneswar, India.

<sup>3</sup> Department of Mathematics, Berhampur University, Bhanja Vihar, Berhampur-760007,  
India.

<sup>4</sup> S.B.R.W. college (Autonomous), Brahmapur-760-001, India.

<sup>5</sup> NIST, Berhampur-761-008, India.

Received: 24 September, 2018 / Accepted: 21 January, 2019 / Published online: 01 May, 2019

**Abstract.** The Lipschitz class of functions was introduced by McFadden while Zygmund developed the method of trigonometric approximation of periodic functions and their Fourier series. Recently, researchers have established many results on different product summability transformations for approximating Fourier series and their conjugate series of periodic functions of different Lipschitz classes. However, in the present article we established a result for approximating conjugate series of a signal of class  $Lip(\beta, p)$  by the product of Euler and matrix summability.

**AMS (MOS) Subject Classification Codes:** 40C05, 40G05.

**Key Words:**  $A$ -mean,  $(E, r)$ -mean, Conjugate series,  $Lip(\beta, p)$ -class, Signal approximation.

## 1. INTRODUCTION

Determination of trigonometric approximation of signals has a rich history associated with several names of mathematicians in the field of theory of summability. Authors like Alexits [1], Bernstein [3], Chandra [4,5], Sahney and Goel [37] and several others have determined the trigonometric approximation of functions of Lipschitz class and their Fourier series by Nörlund and Cesàro transformations of their Fourier series. Subsequently, dealing with functions of class  $Lip \alpha$ ,  $0 < \alpha \leq 1$ , for  $p \geq 1$ ,  $Lip(\alpha, p)$ ,  $Lip(\xi(\tau), p)$ , and  $\mathcal{W}(L_r, \xi(\tau))$  classes, several researchers like Khan [8], Qureshi [36], Rhoades [35] and others established many results on different summability methods for approximating the functions. By the end of twentieth century many researchers such as Khan [9-11], Mittal et

al. [30] and Misra et al. [14] studied degree of trigonometric approximation of functions of various classes using different types of summability transformations. Later, using Nörlund and generalized Nörlund means researchers like Khan [8, 10], Lal and Nigam [13] studied for finding approximations of signals of  $Lip(\beta, p)$  as well as  $\mathcal{W}(L_r, \xi(x))$  classes. Following these results, using linear operators Mishra et al. [24, 25] and Mishra [26] studied approximating signals of class  $\mathcal{W}(L_p, \xi(x))$ . There after using product summability means  $(E, s)(C, 1)$ , Nigam and Sharma [31] estimated the degree of conjugate series of a signal of class  $\mathcal{W}(\xi(x), p)$ .

Working in this direction, Nigam and Sharma [31, 32], Khatri et al. [12], Chandra [6] and V. Mishra et al. [27-29] have also studied the approximation of functions of different classes using different product summability methods and their applications.

Following this, M.Misra et al. [15-21] and U.K.Misra et al. [22, 23] also have established certain results on different product summability methods. In the present study, we continued the work in a similar direction by extending the work of Padhy et al. [33] and obtained a result on the product mean  $(E, s)A$ .

## 2. NOTATIONS AND PRELIMINARIES

Let  $\sum a_n$  be a series and the sequence  $\{s_n\}$  its partial sums. If  $A = (a_{mn})_{\infty \times \infty}$  is a lower triangular matrix, then the transformation

$$t_n = \sum_{i=0}^n a_{ni} s_i, \quad n \in \mathbb{N} \quad (2.1)$$

represents the  $A$ -mean of the sequence  $\{s_n\}$ . Further, if

$$t_n \rightarrow s, \text{ as } n \rightarrow \infty \quad (2.2)$$

then we say that the series  $\sum a_n$  is matrix summable to  $s$  [34].

It is known that matrix summability or  $A$ -summability is regular if and only if the following conditions are satisfying [34]

- (i)  $\sup_m \sum_{n=0}^{\infty} |a_{mn}| < K$ , where  $K$  is an absolute constant
- (ii)  $\lim_{m \rightarrow \infty} a_{mk} = 0$ , for every  $k = 1, 2, \dots$
- (iii)  $\lim_{m \rightarrow \infty} \sum_{n=0}^{\infty} a_{mn} = 1$ .

For any sequence  $\{s_n\}$ , the sequence  $\tau_n$  defined by the transformation

$$\tau_n = \frac{1}{(1+r)^n} \sum_{v=0}^n \binom{n}{v} r^{n-v} s_v, \quad r > 0 \quad (2.3)$$

is called the  $(E, r)$  mean of  $\{s_n\}$  [39]. The series  $\sum a_n$  is summable to  $s$  by  $(E, r)$  method, if

$$\lim_{n \rightarrow \infty} \tau_n = s. \quad (2.4)$$

It is also known,  $(E, r)$  method is regular [7]. Now we define the sequence of  $(E, r)$ -transform of the sequence  $\{t_n\}$  of matrix transform of  $\{s_n\}$  as follows:

$$T_n = \frac{1}{(1+r)^n} \sum_{i=0}^n \binom{n}{i} r^{n-i} t_i = \frac{1}{(1+r)^n} \sum_{i=0}^n \binom{n}{i} r^{n-i} \left\{ \sum_{v=0}^i a_{iv} s_v \right\} \quad (2.5)$$

The series  $\sum a_n$  is summable to  $s$  by the  $(E, r)A$ -summability method, if

$$T_n \rightarrow s, \text{ as } n \rightarrow \infty, \quad (2.6)$$

where  $A = (a_{mn})$  is lower triangular.

For a  $2\pi$  periodic signal  $f(t)$  which is integrable in the sense of Lebesgue over  $(-\pi, \pi)$ , the series

$$\sum_{k=0}^{\infty} A_k(x) \equiv \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \quad (2.7)$$

is called Fourier series of the signal  $f$  at  $x$  and

$$\sum_{k=1}^{\infty} B_k(x) \equiv \sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx), \quad (2.8)$$

$$\text{where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx$$

is called its conjugate series.

**2.1. Definition-1.** The  $\mathcal{L}_{\infty}$ -norm, of the function  $f : R \rightarrow R$  usually denoted by  $\|f\|_{\infty}$ , is

$$\|f\|_{\infty} = \sup_{x \in R} |f(x)| \quad (2.9)$$

and  $\mathcal{L}_v$ -norm, usually denoted as  $\|f\|_v$ , defined over  $[0, 2\pi]$  is defined as

$$\|f\|_v = \left( \int_0^{2\pi} |f(x)|^v dx \right)^{\frac{1}{v}}, \quad v \geq 1. \quad (2.10)$$

**2.2. Definition-2.** The degree of approximation of a real function  $f$  by a trigonometric polynomial  $P_k(x)$  of degree  $k$  under the norm  $\|\cdot\|_{\infty}$  is given by

$$\|P_k - f\|_{\infty} = \sup_{x \in R} |P_k(x) - f(x)| \quad (2.11)$$

and the degree of approximation  $E_k(f)$  of a function  $f \in \mathcal{L}_v$  is defined by

$$E_k(f) = \min_{P_k} \|P_k - f\|_v. \quad (2.12)$$

**2.3. Definition-3.** A real valued signal  $f$  is of Lipschitz class usually denoted by  $f \in Lip \alpha$ , if for  $0 < \alpha \leq 1$

$$|f(y + \theta) - f(y)| = O(|\theta|^\alpha). \quad (2.13)$$

If

$$\left( \int_0^{2\pi} |f(y + \theta) - f(y)|^p dx \right)^{\frac{1}{p}} = O(|\theta|^\beta), \quad (2.14)$$

where  $\theta > 0, 0 < y \leq 2\pi, p \geq 1, 0 < \beta \leq 1$  then we say  $f \in Lip(\beta, p)$ .

**2.4. Notations.** Throughout this article the following notations will be used

$$\psi(t) = \frac{1}{2} \{f(y + t) - f(y - t)\}, \quad (2.15)$$

$$\bar{S}_m(f; x) = \sum_{k=1}^m B_k(x) \quad (2.16)$$

$$\bar{r}_m(t) = \frac{1}{\pi(r+1)^m} \sum_{k=0}^m \binom{m}{k} r^{m-k} \left\{ \sum_{v=0}^k a_{kv} \frac{\cos \frac{t}{2} - \cos(v + \frac{1}{2})t}{\sin \frac{t}{2}} \right\} \quad (2.17)$$

$$(E, q)A - \text{method is regular, where } A \text{ is a matrix.} \quad (2.18)$$

### 3. KNOWN RESULTS

In 2010, Nigam and Sharma [32] proved

**Theorem 3.1.** If the signal  $f$  is  $2\pi$ -periodic and of class  $Lip\beta$ , then its approximation by  $(E, s)(C, 1)$  method, of Fourier series of  $\sum_{n=0}^{\infty} A_n(t)$  of  $f$  is given by

$$\|E_n^s C_n^1 - f\|_{\infty} = O\left(\frac{1}{(1+n)^\beta}\right), \quad 0 < \beta < 1,$$

where  $E_n^s C_n^1$  represents  $(E, s)$  mean of  $(C, 1)$  mean of  $S_n(f; x)$ .

For the function  $f \in Lip(\alpha, l)$ ,  $l \geq 1$ , Padhy et.al. [33] established a result using  $(E, s)A$ -method of Fourier series of  $f$ .

**Theorem 3.2.** If the signal  $f$  is  $2\pi$ -periodic and of class  $Lip(\beta, l)$ , then approximation by  $(E, s)A$  method, of the Fourier series of  $f$  satisfies  $\|T_n - f\|_{\infty} = O\left(\frac{1}{(1+n)^{\beta-\frac{l}{t}}}\right)$ ,  $0 < \beta < 1, l \geq 1$ , where  $T_n$  is as defined in (2.5).

In the next section, we have established a parallel theorem for conjugate series, which is our main result.

## 4. MAIN THEOREM

**Theorem 4.1.** If the signal  $f$  is  $2\pi$ -periodic and of class  $Lip(\beta, l)$ , then the approximation by  $(E, s)A$ -method of the conjugate series of  $f$  is of order  $\|T_n - f\|_\infty = O\left(\frac{1}{(1+n)^{\beta-\frac{1}{l}}}\right)$ ,  $0 < \beta < 1$ ,  $l \geq 1$ , where  $T_n$  is as defined in (2.5) and  $A$  is a lower-triangular matrix.

## 5. REQUIRED LEMMA

The following lemma is necessary for our proof:

**Lemma 5.1.**

$$|\bar{\kappa}_n(t)| = \begin{cases} O(n), & 0 \leq t \leq \frac{1}{1+n} \\ O(\frac{1}{t}), & \frac{1}{1+n} \leq t \leq \pi \end{cases}$$

The proof of the Lemma is in [32].

## 6. PROOF OF THEOREM 4.1

Using Riemann-Lebesgue theorem we get

$$\bar{S}_n(f; x) - f(x) = \frac{2}{\pi} \int_0^\pi \psi(u) \frac{\cos \frac{u}{2} - \cos(n + \frac{1}{2})u}{2 \sin \frac{u}{2}} du$$

Following Titchmarsh [38] and using (2.1), the  $A$ -transform of  $n$ -th partial sum of the conjugate series of  $f(x)$ , we get

$$u_n - f(x) = \frac{2}{\pi} \int_0^\pi \psi(u) \sum_{k=0}^n a_{nk} \frac{\cos \frac{u}{2} - \cos(k + \frac{1}{2})u}{2 \sin \frac{u}{2}} du,$$

Denoting  $(E, s)A$  transformation of the  $n$ -th partial sum by  $\tau_n$ , we get the

$$\begin{aligned} |T_n - f(x)| &= \frac{2}{\pi(1+s)^n} \int_0^\pi \psi(u) \sum_{i=0}^n \binom{n}{i} r^{n-i} \left\{ \sum_{v=0}^i a_{iv} \frac{\cos \frac{u}{2} - \cos(v + \frac{1}{2})u}{2 \sin \frac{u}{2}} \right\} du \\ &= \int_0^\pi \psi(u) \bar{\kappa}_n(u) du \\ &= \int_0^{\frac{1}{1+n}} \psi(u) \bar{\kappa}_n(u) du + \int_{\frac{1}{1+n}}^\pi \psi(u) \bar{\kappa}_n(u) du \\ (A) \quad &= I_1 + I_2 \end{aligned}$$

Now, we have

$$|I_1| \leq \left| \int_0^{\frac{1}{1+n}} \psi(u) \bar{\kappa}_n(u) du \right|$$

Using Hölder's inequality, by taking  $\frac{1}{l} + \frac{1}{m} = 1$ , we get

$$\begin{aligned}
|I_1| &= \left( \int_0^{\frac{1}{1+n}} |\psi(u)|^l dt \right)^{\frac{1}{l}} \left( \int_0^{\frac{1}{1+n}} |\bar{\kappa}_n(u)|^m du \right)^{\frac{1}{m}} \\
&= O \left( \frac{1}{(1+n)^\beta} \right) \left( \int_0^{\frac{1}{1+n}} n^m du \right)^{\frac{1}{m}} \\
&= O \left( \frac{1}{(1+n)^\beta} \right) \left( \frac{n^m}{1+n} \right)^{\frac{1}{m}} \\
&= O \left( \frac{1}{(1+n)^{\frac{1}{m}-1+\beta}} \right) \\
(B) \quad &= O \left( \frac{1}{(1+n)^{\beta-\frac{1}{l}}} \right)
\end{aligned}$$

Next, similarly to the estimation of  $|I_1|$ , we obtain with the help of Lemma - 5.1

$$\begin{aligned}
|I_2| &\leq \int_{\frac{1}{1+n}}^{\pi} |\psi(u)| |\bar{\kappa}_n(u)| du \\
&= \left( \int_{\frac{1}{1+n}}^{\pi} |\psi(u)|^l du \right)^{\frac{1}{l}} \left( \int_{\frac{1}{1+n}}^{\pi} |\bar{\kappa}_n(u)|^m du \right)^{\frac{1}{m}} \\
&= O \left( \frac{1}{(1+n)^\beta} \right) \left( \int_{\frac{1}{1+n}}^{\pi} \left( \frac{1}{u} \right)^m du \right)^{\frac{1}{m}} \\
&= O \left( \frac{1}{(1+n)^\beta} \right) \left( [u^{-m+1}]_{\frac{1}{1+n}}^{\pi} \right)^{\frac{1}{m}} \\
&= O \left( \frac{1}{(1+n)^\beta} \right) \left( \frac{1}{1+n} \right)^{\frac{1-m}{m}} \\
&= O \left( \frac{1}{(1+n)^{\beta-1+\frac{1}{m}}} \right) \\
(C) \quad &= O \left( \frac{1}{(1+n)^{\beta-\frac{1}{l}}} \right)
\end{aligned}$$

Now from (B) and (C), we get

$$|T_n - f(x)| = O \left( \frac{1}{(1+n)^{\beta-\frac{1}{l}}} \right), 0 < \beta < 1, l \geq 1$$

Therefore,

$$\|T_n - f(x)\| = \sup_{-\pi < x < \pi} O \left( \frac{1}{(1+n)^{\beta-\frac{1}{l}}} \right) = O \left( \frac{1}{(1+n)^{\beta-\frac{1}{l}}} \right), 0 < \beta < 1, l \geq 1.$$

This ends the proof.

## 7. ACKNOWLEDGEMENT

The authors wish to express their sincere gratitude to the anonymous referees for careful reading and giving constructive comments and helpful suggestions to improve the presentation and readability of the paper.

## REFERENCES

- [1] G. Alexits, *Über die Annäherung einer stetigen function durch die Cesàrochen Mittel in hrer Fourier reihe*, Math. Annal. **100**, (1928) 264-277.
- [2] G. Alexits, *Convergence problems of orthogonal series*, Pergamon press, London, 1961.
- [3] S. Bernstein, *Sur L'ordre de la meilleure approximation des fonctions par des polynomes de degre donne*, Mem. Cl. Sci. Acad. Roy. Belg. **4**, (1912) 1-103.
- [4] P. Chandra, *On degree of Approximation of functions belonging to Lipschitz class*, Nanta Math. **80**, (1975) 88-89.
- [5] P. Chandra, *Approximation by Nörlund operators*, Math. Vestnik, **38**, (1986) 263-269.
- [6] P. Chandra, *Trigonometric approximation of functions in  $L_p$  norm*, J. Math. Anal. Appl. **275**, (2002) 13-26.
- [7] G. H. Hardy, *Divergent series*, First edition, Oxford University press, 1949.
- [8] H. H. Khan, *On the degree of approximation of a function belonging to the class  $Lip(\alpha, p)$* , Indian J. Pure Appl. Math. **5**, (1974) 132-136.
- [9] H. H. Khan, *Approximation of classes of functions*, Ph.D Thesis, AMU Aligarh, 1974.
- [10] H. H. Khan, *On the degree of approximation to a function belonging to weighted  $W(L_p, \xi(t))$  ( $p \geq 1$ ) class*, The Aligarh Bull. Math. **3-4**, (1973-74) 83-88.
- [11] H. H. Khan, *A note on a theorem of Izumi*, Comm. Fac. Maths. Ankara, **31**, (1982) 123-127.
- [12] K. Khatri, V. N. Mishra and H. M. Srivastava, *Approximation of functions belonging to  $L(0, \infty)$  by product summability means of its Fourier-Laguerre series*, Cogent Mathematics, **3**, (2016) 1-7.
- [13] S. Lal and H. K. Nigam, *Degree of approximation of conjugate of a function belonging to  $Lip(\xi(t), p)$  class by matrix summability means of conjugate Fourier series*, Int. J. of Math. and Mathematical Sciences, **27**, No. 9 (2001) 555-563.
- [14] L. N. Misra, V. N. Mishra, K. Khatri and Deepmala, *On the trigonometric approximation of signals(functions) belonging to generalized weighted Lipschitz  $W(L^r, \xi(t))$  ( $r \geq 1$ )-class by matrix  $(C, N_p)$  operator of conjugate series of its Fourier series*, Applied Mathematics and Computation **237**, (2014) 252-263.
- [15] M. Misra, B. P. Padhy, P. Samanta, P. K. Das and U.K. Misra, *Trigonometric approximations of conjugate series of a function of Generalized Lipschitz class by product summability*, Journal of Orissa Mathematical Society **23**, No. 2 (2014) 124-132.
- [16] M. Misra, B. P. Padhy, P. Samanta, B. Majhi and U. K. Misra, *Approximation of conjugate series of the Fourier series of the class  $W(L^p, \xi(t))$  by product mean*, American Journal of Applied Mathematics and Statistics **2**, No. 5 (2014) 352-356.
- [17] M. Misra, B. P. Padhy, S. Nayak and U. K. Misra, *Degree of Approximation of conjugate series of a Fourier series by product  $(E, R)(N, p, q)$  summability methods*, Journal of Advances in Mathematics **9**, No. 2 (2014) 1955-1962.
- [18] M. Misra, B. P. Padhy, D. Bisoyi and U. K. Misra, *On Degree of Approximation of conjugate series of a Fourier series by product summability*, Malaya Journal of Mathematics **1**, No. 1 (2013) 37-42.
- [19] M. Misra, B. P. Padhy, N. Panda and U. K. Misra, *Approximation of conjugate series of a Fourier series by product summability*, Journal of Ultra Scientist of Physical Sciences **24**, No. 3A (2012), 501 -508 .
- [20] M. Misra, B. P. Padhy, S. K. Buxi and U. K. Misra, *On Degree of Approximation by product means of conjugate series of Fourier series*, International Journal of Math. Science and Engineering applications **6**, No. 1 (2012) 363 - 370 .
- [21] M. Misra, B. P. Padhy, B. Mallick and U. K. Misra, *On Degree of Approximation of conjugate series of Fourier series by product mean*, International Arcive of Applied Sciences **2**, No. 2 (2011) 31 - 37.
- [22] U. K. Misra, S. K. Paikray, R. K. Jati and N. C. Sahoo, *On Degree of Approximation by product means of conjugate series of Fourier series*, Bulletin of Society for Mathematical Services and standards **1**, No. 4 (2012) 12 - 20 .

- [23] U. K. Misra, S. Sahu, D. Acharya and P. Nayak, *Trigonometric Approximation of conjugate series of a function of generalized Lipschitz class by product summability*, AIP Conference Proceeding, 1751.020005, (2016) doi: 10.1063/1.4954858.
- [24] V. N. Mishra, K. Khatri and L. N. Misra, *Product summability transform of conjugate series of Fourier series*, International J. of Math. and Mathematical Sciences, Article ID-298923(2012), 13 pages.
- [25] V. N. Mishra, K. Khatri and L. N. Misra, *Approximation of functions belonging to  $Lip(\xi(t), r)$ -class by  $(N, p_n)(E, q)$  summability of Conjugate series of Fourier series*, Journal of Inequalities and Applications, (2012) 296.
- [26] V. N. Mishra, *On the Degree of Approximation of signals(functions) belonging to Generalized weighted  $W(L_p, \xi(t))$  ( $p \geq 1$ )-class by almost matrix summability method of its conjugate Fourier series*, Int. J. of Appl. Math. and Mech. **5**, No. 7 (2009) 16-27.
- [27] V. N. Mishra, K. Khatri, L. N. Misra and Deepmala, *Trigonometric approximation of signals belonging to generalized weighted Lipschitz  $W(L_r, \xi(t))$ , ( $r \geq 1$ )-class by matrix  $(N, p_n)(E, q)$  operator of conjugate series of its Fourier series*, Journal of Classical Analysis **5**, No. 2 (2014) 91-105.
- [28] V. N. Mishra, H. Khan, K. Khatri and L. N. Misra, *Degree of approximation of conjugate of signals(functions) belonging to the generalized weighted Lipschitz  $W(L_r, \xi(t))$ , ( $r \geq 1$ )-class by  $(C, 1)(E, q)$  means of conjugate trigonometric Fourier series*, Bulletin of Mathematical Analysis and Applications **5**, No. 4 (2013) 40-53.
- [29] V. N. Mishra, H. H. Khan, K. Khatri and L. N. Misra, *On approximation of conjugate signals(functions) belonging to the generalized weighted  $W(L_r, \xi(t))$ , ( $r \geq 1$ ) class by product summability means of Conjugate series of Fourier series*, International Journal of Mathematical Analysis **6**, No. 35 (2012) 1703-1715.
- [30] M. L. Mittal, B. E. Rhoades, V. N. Mishra, S. Priti and S. S. Mittal, *Approximation of functions belonging to  $Lip(\xi(t), p)$ ,  $p \geq 1$  class by means of conjugate Fourier series using linear operators*, Indian J. of Math. **47**, (2005) 217-229.
- [31] H. K. Nigam and A. Sharma, *On approximation of conjugate of a function belonging to  $Lip(x(t), r)$ -class by product summability means of conjugate series of Fourier series*, Int. J. of Contem. Math. Sci. **5**, No. 54 (2010) 2673-2683.
- [32] H. K. Nigam and A. Sharma, *On degree of approximation by product means*, Ultra Scientist of Physical Sciences **22**, No. 3 (2010) 889-894.
- [33] R. Ch. Padhy, B. P. Padhy, P. N. Samanta, M. Misra and U. K. Misra, *A theorem on degree of approximation of Fourier series of a function of Lipschitz class by product means*, Int. J. of Math. and Computation, **29**, No. 3 (2018) 71-78.
- [34] G. M. Petersen, *Regular Matrix Transformation*, McGraw-Hill Publishing company Limited, 1966.
- [35] B. E. Rhoades, *On the degree of approximation of functions belonging to Lipschitz class by Hausdorff means of its Fourier series*, Tamkang J. Math. **34**, (2003) 245-247.
- [36] K. Qureshi, *On the degree of approximation of a function belonging to weighted  $W(L(r), \zeta(t))$  class*, Indian J. of Pure Appl. Math. **13**, (1982) 471-475.
- [37] B. N. Sahney and D. S. Goel, *On the degree of continuous functions*, Ranchi University, Math. Jour. **4**, (1973) 50-53.
- [38] E. C. Titchmarsh, *The Theory of functions*, Oxford University Press, 2nd edition (1952) 402-403.
- [39] A. Zygmund, *Trigonometric Series*, Second edition, Cambridge University Press, Cambridge 1959.