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Designing Shapes and Handling Noisy Data with Weighted Least Squares-Based Subdivision Schemes

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Abstract. This article explores the application of the weighted least squares (WLS) method in the development of subdivision schemes. These schemes offer a valuable means of approximating data points, whether they belong to linear or non-linear functions. Moreover, the schemes demonstrate effective handling of noisy data with outliers during the fitting process. Additionally, they prove to be highly effective in generating engineering shapes. In comparison to the traditional least squares (LS) approximation, WLS provides a more flexible approach. The resulting subdivision schemes produced by WLS are capable of generating smooth curves and surfaces without explicit function representations. In this work, generalized WLS-based methods for the creation of stationary and non-stationary subdivision schemes are presented. Furthermore, it includes a comparison between LS and WLS-based schemes, highlighting that the schemes produced by the LS method are a special case of those generated by WLS. By shedding light on the advantages of WLS over LS, this article contributes to the understanding and utilization of WLS-based subdivision schemes in various contexts.

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Key Words: Curves; surfaces; stationary schemes; non-stationary schemes; weighted least squares; noisy data.

1. INTRODUCTION

The process of constructing curves and surfaces from a set of data points is known as curve and surface fitting. Subdivision schemes are popular techniques used for fitting curves and surfaces. These techniques can be categorized as statistical subdivision schemes when generated using statistical methods, and classical subdivision schemes when generated using methods other than statistical approaches. While subdivision curves and surfaces are extensively utilized in geometric modeling. There remains a need for further exploration in certain areas. New subdivision schemes must be developed to address the demands of emerging practical applications [8]. For instance, classical subdivision schemes frequently do not effectively handle noisy data. Further subdivision schemes can be classified into stationary and non-stationary schemes. Nowadays, there are numerous stationary and non-stationary schemes available. Recently, Yang et al. [18] presented a family of four-point stationary subdivision schemes. Conti et al. [1] introduced one of the non-stationary schemes. Daniel and Shunmugaraj [2] introduced stationary and non-stationary schemes. In 2015, Dyn et al. [3] introduced statistical subdivision schemes as a means to approximate noisy data. These schemes utilized least squares polynomials for refining noisy data.

The subdivision schemes still require more effective approaches for handling noisy data. There is always room for improving the efficiency of existing subdivision techniques. The interpolation technique for irregularly spaced points was initially introduced by Shepard [13]. Mutiu [12] presented the application of weighted least squares method. Usman et al. [15] presented the use of WLS when data have error variance is heteroscedastic. Tarrio et al. [14] presented a technique for imporving the singles by using the weighted least squares.

The Weighted Least Squares (WLS) technique has found application in various domains, with different methods relying on its principles. For instance, Liu et al. [7] employed WLS-based non-oscillatory schemes for finite volume methods on unstructured meshes. Wang et al. [16] utilized a learning-based local weighted least squares approach for the algebraic multigrid method. Zhou et al. [19] introduced the strategy of weighted WLS for image reconstruction. Additionally, Huang et al. [5] presented a global-local image enhancement technique with contrast improvement based on WLS. Giordani and Kiers [4] applied WLS for archetypal analysis with missing data. Moreover, Wang et al. [17] also incorporated WLS techniques in their work. The family of binary approximating schemes with error analysis was presented in [11]. The l_1-regression-based subdivision schemes for noisy data were presented by Mustafa et al. [9]. This work is an extension of the work presented in [3]. A new modified least squares method with real life application was presented in [6].

This article proposes a novel approach that unifies and generalizes subdivision schemes including both stationary and non-stationary schemes, within a single framework. The objective is to enable efficient handling of noisy data for fitting purposes and shape design. To achieve this, we benefit from the advantages of weighted least squares algorithms.

The paper is organized in the following manner: Section 2 and Section 3 presents the generalized algorithms of univariate and bivariate case subdivision schemes respectively.

Section 4 conducts a comparison between subdivision schemes based on the WLS method and those based on the LS method. Finally, Section 5 provides the conclusions of the study.

2. The algorithm for curve schemes

The derivation of b-ary univariate stationary and non-stationary schemes involves the following steps:
Step 1: Consider the polynomial

$$f(x_\lambda) = c_0 + c_1 x_\lambda + c_2 x_\lambda^2 + c_3 x_\lambda^3 + \cdots + c_p x_\lambda^p. \tag{2. 1}$$

For the best fit of (2. 1) with observations $(x_\lambda = \lambda, f_\lambda)$ for $\lambda = -\nu, -\nu+1, \cdots, \nu$, where $\nu > 0$. The residual is defined as

$$J(c_0, c_1, c_2, \cdots, c_p) = \sum_{\lambda=-\nu}^{\nu} w_\lambda \left(f_\lambda - \sum_{i=0}^{p} c_i \lambda^i \right)^2, \tag{2. 2}$$

We get the normal equations by minimizing the sum of squares of the residual

$$\sum_{\lambda=-\nu}^{\nu} \lambda^q f_\lambda w_\lambda = \sum_{j=0}^{p} c_j \sum_{\lambda=-\nu}^{\nu} \lambda^{j+q} w_\lambda, \quad q = 0, 1, 2, \cdots, p. \tag{2. 3}$$

In the first step, the normal equations for a polynomial of degree p are derived by substituting the corresponding values of p in equation (2. 3). The coefficients c_0, c_1, $\cdots$, c_p are obtained by solving these normal equations as defined in equation (2. 3). Once the values of c_p's are determined, they are substituted into equation (2. 1) to obtain the best fit polynomial.
Step 2: The b-ary family of schemes is derived by setting $\lambda = \pm\frac{2t+1}{2b}$, where $b \geq 2$ and $t = 0, 1, 2, \cdots, b-1$. In this step, the terms f_λ are replaced by $f_{i+\lambda}^{k}$ and $f(\lambda)$ is replaced by $f_{bi+\alpha}^{k+1}$ in the best fit polynomial of any degree. Similarly, if $-\nu+1 \leq \lambda \leq \nu$, where $\nu \geq 1$, 2ν-point b-ary schemes for curve design can be obtained.

2.1. The 1^{st} degree polynomial and $(2\nu+1)$-point, b-ary schemes.

The linear polynomial is obtained after substituting $p = 1$ in (2.1), the corresponding normal equations obtained from (2.3) are

$$\begin{aligned}
\sum_{\lambda=-\nu}^{\nu} f_\lambda w_\lambda &= c_0 \sum_{\lambda=-\nu}^{\nu} w_\lambda + c_1 \sum_{\lambda=-\nu}^{\nu} \lambda w_\lambda, \\
\sum_{\lambda=-\nu}^{\nu} \lambda f_\lambda w_\lambda &= c_0 \sum_{\lambda=-\nu}^{\nu} \lambda w_\lambda + c_1 \sum_{\lambda=-\nu}^{\nu} \lambda^2 w_\lambda.
\end{aligned}$$

The values of c_0 and c_1 by solving the above equations

$$a_0 = \frac{\xi_2 \eta_0 - \xi_1 \eta_1}{\xi_2 \xi_0 - \xi_1^2} \quad \text{and} \quad a_1 = -\frac{\xi_1 \eta_0 - \xi_0 \eta_1}{\xi_2 \xi_0 - \xi_1^2},$$

where

$$\xi_0 = \sum_{\lambda=-\nu}^{\nu} w_\lambda, \quad \xi_1 = \sum_{\lambda=-\nu}^{\nu} \lambda w_\lambda, \quad \xi_2 = \sum_{\lambda=-\nu}^{\nu} \lambda^2 w_\lambda, \quad \eta_0 = \sum_{\lambda=-\nu}^{\nu} f_\lambda w_\lambda,$$

$$\text{and} \quad \eta_1 = \sum_{\lambda=-\nu}^{\nu} \lambda f_\lambda w_\lambda.$$

After putting the values of c_0 and c_1, the best fit polynomial is obtained

$$f(\lambda) = \frac{\xi_2\eta_0 - \xi_1\eta_1}{\xi_2\xi_0 - \xi_1^2} - \left(\frac{\xi_1\eta_0 - \xi_0\eta_1}{\xi_2\xi_0 - \xi_1^2}\right)\lambda. \tag{2. 4}$$

The $(2\nu+1)$-point b-ary schemes are obtained by putting $\lambda = \pm\frac{2t+1}{2b}$ in (2.4) and replacing f_λ by $f^k_{i+\lambda}$ and $f(\lambda) = f^{k+1}_{bi+\alpha}$

$$f^{k+1}_{bi+\alpha} = f\left(\pm\frac{2t+1}{2b}\right) = \frac{\xi_2\eta_0 - \xi_1\eta_1}{\xi_2\xi_0 - \xi_1^2} - \left(\frac{\xi_2\eta_0 - \xi_1\eta_1}{\xi_2\xi_0 - \xi_1^2}\right)\left(\pm\frac{2t+1}{2b}\right),$$
$$\alpha = 0, 1 \cdots, b-1, \tag{2. 5}$$

where

$$\xi_0 = \sum_{\lambda=-\nu}^{\nu} w_\lambda, \quad \xi_1 = \sum_{\lambda=-\nu}^{\nu} \lambda w_\lambda, \quad \xi_2 = \sum_{\lambda=-\nu}^{\nu} \lambda^2 w_\lambda, \quad \eta_0 = \sum_{\lambda=-\nu}^{\nu} f^k_{i+\lambda} w_\lambda,$$

$$\text{and} \quad \eta_1 = \sum_{\lambda=-\nu}^{\nu} \lambda f^k_{i+\lambda} w_\lambda.$$

Equation (2. 5) represents the family of odd points b-ary schemes. The specific schemes within this family are obtained after putting the values of ν and b. The choice of ν determines the complexity of the scheme, with different values leading to different numbers of points in the scheme. For instance, when ν takes values of 1, 2, and 3, we obtain 3-point, 5-point, and 7-point schemes, respectively. Similarly, by varying the value of b, we obtain schemes with different arities. For example, for b equal to 2, 3, and 4, we have binary, ternary, and quaternary schemes, respectively. The flexibility to adjust ν and b allows for the generation of a wide range of schemes with varying complexities and arities.

2.1.1. *Derivation of stationary schemes.* In this section, we provide examples of some family members of stationary schemes. By considering the weight function $w_\lambda = \frac{1}{|x-\lambda|}$ with $x = \frac{1}{4}$ and setting $b = 2$ in equation (2. 5), we obtain the $(2\nu+1)$-point binary scheme denoted as $M_{2\nu+1}$. Specifically, when $\nu = 1$, we obtain the 3-point scheme denoted as M_3.

$$\begin{aligned} f^{k+1}_{2i} &= \frac{1}{16}\left(5f^k_{i-1} + 10f^k_i + f^k_{i+1}\right), \\ f^{k+1}_{2i+1} &= \frac{1}{16}\left(f^k_{i-1} + 10f^k_i + 5f^k_{i+1}\right). \end{aligned} \tag{2. 6}$$

When $\nu = 2$ in (2.5), we get a 5-point scheme M_5

$$\begin{aligned} f_{2i}^{k+1} &= \frac{1}{688}\left(72f_{i-2}^k + 147f_{i-1}^k + 378f_i^k + 63f_{i+1}^k + 28f_{i+2}^k\right), \\ f_{2i+1}^{k+1} &= \frac{1}{688}\left(28f_{i-2}^k + 63f_{i-1}^k + 378f_i^k + 147f_{i+1}^k + 72f_{i+2}^k\right). \end{aligned} \tag{2. 7}$$

For even point schemes, we consider values of λ as $\lambda = -\nu+1, -\nu+2, \cdots, \nu$, which results in schemes with C^1 continuity. By using the weight function $w_\lambda = e^{-\frac{|x-\lambda|}{100}}$ with $x = \frac{1}{4}$ and setting $b = 2$ in equation (2. 5), we obtain the $(2\nu+1)$-point scheme denoted as $M'_{2\nu+1}$. Specifically, when considering the 3-point scheme, denoted as M'_3, we substitute $\nu = 1$ in equation (2. 5).

$$\begin{aligned} f_{2i}^{k+1} &= 0.4576032960f_{i-1}^k + 0.3347934077f_i^k + 0.207603296f_{i+1}^k, \\ f_{2i+1}^{k+1} &= 0.207603296f_{i-1}^k + 0.3347934077f_i^k + 0.4576032960f_{i+1}^k. \end{aligned} \tag{2. 8}$$

For the 5-point scheme M'_5 put $\nu = 2$ in (2.5)

$$\begin{aligned} f_{2i}^{k+1} &= 0.2483505598f_{i-2}^k + 0.2258847266f_{i-1}^k + 0.201930049f_i^k \\ &\quad +0.1750834822f_{i+1}^k + 0.148751182f_{i+2}^k, \\ f_{2i+1}^{k+1} &= 0.148751182f_{i-2}^k + 0.1750834822f_{i-1}^k + 0.201930049f_i^k \\ &\quad +0.2258847266f_{i+1}^k + 0.2483505598f_{i+2}^k. \end{aligned} \tag{2. 9}$$

The aforementioned schemes, which are generated based on the weighted least squares method, produce comparable results presented by Dyn et al. [3] when dealing with noisy data that contains outliers. This can be observed in Figure 2.

2.1.2. *Derivation of non-stationary schemes.* By utilizing the weight function $w_\lambda = \cos(\frac{x-\lambda}{2^k})$ (where x is a fixed point) and setting $b = 2$ in equation (2.5), we obtain the $(2\nu+1)$-point non-stationary scheme denoted as $M_{2\nu+1}^k$. To derive the specific 3-point non-stationary scheme, denoted as M_3^k, we substitute $\nu = 1$ into equation (2.5).

$$\begin{aligned} f_{2i}^{k+1} &= \frac{1}{4\eta_4^k}\Big(\eta_3^k f_{i-1}^k + \eta_2^k f_i^k + \eta_1^k f_{i+1}^k\Big), \\ f_{2i+1}^{k+1} &= \frac{1}{4\eta_4^k}\Big(\eta_1^k f_{i-1}^k + \eta_2^k f_i^k + \eta_3^k f_{i+1}^k\Big), \end{aligned} \tag{2. 10}$$

where

$$\begin{aligned} \eta_1^k &= 6\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{3}{4.2^k}\right) - \cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right), \\ \eta_2^k &= 5\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right) + 3\cos\left(\frac{3}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right), \\ \eta_3^k &= 10\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{3}{4.2^k}\right) + \cos\left(\frac{3}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right), \end{aligned}$$

and

$$\begin{aligned}\eta_4^k &= \cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right)+4\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{3}{4.2^k}\right)\\&\quad+\cos\left(\frac{3}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right).\end{aligned}$$

For a 5-point non-stationary scheme M_5^k put $\nu=2$ in (2.5),

$$\begin{aligned}f_{2i}^{k+1}&=\frac{-1}{4\eta_{10}^k}\Big(\eta_9^k f_{i-2}^k+\eta_8^k f_{i-1}^k+\eta_7^k f_i^k+\eta_6^k f_{i+1}^k+\eta_5^k f_{i+2}^k\Big),\\f_{2i+1}^{k+1}&=\frac{-1}{4\eta_{10}^k}\Big(\eta_5^k f_{i-2}^k+\eta_6^k f_{i-1}^k+\eta_7^k f_i^k+\eta_8^k f_{i+1}^k+\eta_9^k f_{i+2}^k\Big),\end{aligned}\qquad (2.\ 11)$$

where

$$\begin{aligned}\eta_5^k &= 2\cos\left(\frac{9}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right)-9\cos\left(\frac{9}{4.2^k}\right)\cos\left(\frac{3}{4.2^k}\right)\\&\quad+5\cos\left(\frac{9}{4.2^k}\right)\cos\left(\frac{5}{4.2^k}\right)-28\cos\left(\frac{9}{4.2^k}\right)\cos\left(\frac{7}{4.2^k}\right),\\
\eta_6^k &= -9\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{9}{4.2^k}\right)-21\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{7}{4.2^k}\right)\\&\quad+\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right)-6\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{3}{4.2^k}\right),\\
\eta_7^k &= -18\cos\left(\frac{9}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right)-3\cos\left(\frac{3}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right)\\&\quad-5\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right)-14\cos\left(\frac{7}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right),\\
\eta_8^k &= -27\cos\left(\frac{9}{4.2^k}\right)\cos\left(\frac{3}{4.2^k}\right)-10\cos\left(\frac{3}{4.2^k}\right)\cos\left(\frac{5}{4.2^k}\right)\\&\quad-7\cos\left(\frac{3}{4.2^k}\right)\cos\left(\frac{7}{4.2^k}\right)-\cos\left(\frac{3}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right),\\
\eta_9^k &= 3\cos\left(\frac{3}{4.2^k}\right)\cos\left(\frac{7}{4.2^k}\right)-2\cos\left(\frac{7}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right)\\&\quad-15\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{7}{4.2^k}\right)-36\cos\left(\frac{9}{4.2^k}\right)\cos\left(\frac{7}{4.2^k}\right),\end{aligned}$$

and

$$\begin{aligned}
\eta_{10}^{k} \quad = \quad & \cos\left(\frac{9}{4.2^k}\right)\cos\left(\frac{5}{4.2^k}\right) + 4\cos\left(\frac{9}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right) \\
& +9\cos\left(\frac{9}{4.2^k}\right)\cos\left(\frac{3}{4.2^k}\right) + 16\cos\left(\frac{9}{4.2^k}\right)\cos\left(\frac{7}{4.2^k}\right) \\
& +\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right) + 4\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{3}{4.2^k}\right) \\
& +9\cos\left(\frac{5}{4.2^k}\right)\cos\left(\frac{7}{4.2^k}\right) + \cos\left(\frac{3}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right) \\
& +\cos\left(\frac{3}{4.2^k}\right)\cos\left(\frac{7}{4.2^k}\right) + 4\cos\left(\frac{7}{4.2^k}\right)\cos\left(\frac{1}{4.2^k}\right).
\end{aligned}$$

Additionally, we can generate the 2ν-point non-stationary counterpart of the schemes introduced in [3] by choosing $\lambda = -\nu+1, -\nu+2, \cdots, \nu$. The asymptotical equivalence, as established in [1], is employed to demonstrate that the non-stationary schemes correspond to the stationary schemes presented in [3].

Proposition 2.2. *The stationary scheme in* [3]*, characterized by the mask* $\frac{1}{24}[5, 11, 8, 8, 11, 5]$*, exhibits asymptotic equivalence to the mask of the non-stationary scheme described in* (2.10).

Proof
From the equivalence relation, we get

$$\lim_{k\to+\infty} \eta_1^k = 5, \quad \lim_{k\to+\infty} \eta_2^k = 8, \quad \lim_{k\to+\infty} \eta_3^k = 11, \quad \text{and} \quad \lim_{k\to+\infty} \eta_4^k = 6.$$

The non-stationary schemes (2.10) takes the form

$$\begin{aligned}
f_{2i}^{k+1} &= \frac{1}{24}\Big(11f_{i-1}^k + 8f_i^k + 5f_{i+1}^k\Big), \\
f_{2i+1}^{k+1} &= \frac{1}{24}\Big(5f_{i-1}^k + 8f_i^k + 11f_{i+1}^k\Big).
\end{aligned}$$

The mask of the aforementioned scheme is indeed $\frac{1}{24}[5, 11, 8, 8, 11, 5]$. Which completes the proof.

Proposition 2.3. *The stationary scheme presented in* [3] *with the mask* $\frac{1}{40}[6, 10, 7, 9, 8, 8, 9, 7, 10, 6]$ *is asymptotically equivalent to the mask of the non-stationary scheme described in* (2.11).

Proof
From the equivalence relation, we can conclude that

$$\begin{aligned}
&\lim_{k\to+\infty} \eta_5^k = -30, \quad \lim_{k\to+\infty} \eta_6^k = -35, \quad \lim_{k\to+\infty} \eta_7^k = -40, \\
&\lim_{k\to+\infty} \eta_8^k = -45, \quad \lim_{k\to+\infty} \eta_9^k = -50, \quad \text{and} \quad \lim_{k\to+\infty} \eta_{10}^k = 50.
\end{aligned}$$

The non-stationary scheme (2.11) takes the form

$$f_{2i}^{k+1} = \frac{1}{40}\Big(10f_{i-2}^{k} + 9f_{i-1}^{k} + 8f_{i}^{k} + 7f_{i+1}^{k} + 6f_{i+2}^{k}\Big),$$
$$f_{2i+1}^{k+1} = \frac{1}{40}\Big(6f_{i-2}^{k} + 7f_{i-1}^{k} + 8f_{i}^{k} + 9f_{i+1}^{k} + 10f_{i+2}^{k}\Big).$$

The mask of the above scheme is $\frac{1}{40}[6, 10, 7, 9, 8, 8, 9, 7, 10, 6]$. This confirms the completion of the proof.

Remark 2.4. *The proposed algorithm is capable of reproducing existing least squares-based subdivision schemes.*

- *The odd point family* [3] *reproduced after putting $w_\lambda = 1$ and $b = 2$ in (2. 5).*
- *The even point family* [3] *reproduced after putting $w_\lambda = 1$, $b = 2$ and ensuring $-\nu + 1 \leq \lambda \leq \nu$ in (2. 5).*
- *The family of odd point, b-ary schemes* [10] *reproduced after setting $w_\lambda = 1$ and $p = 3$, in Section 2.*
- *The family of odd point, b-ary schemes* [10] *reproduced after setting $w_\lambda = 1$, $p = 3$ and ensuring $-\nu + 1 \leq \lambda \leq \nu$ in Section 2.*

3. THE ALGORITHM FOR SURFACE SCHEMES

The immediate extension of subdivision schemes from curve modeling to surface modeling involves the transition from univariate to bivariate schemes. In this section, we employ the WLS algorithm to introduce stationary and non-stationary tensor product schemes with various arities.

Step 1: Consider a bivariate polynomial with respect to the observations $(x_\lambda = \lambda, y_\delta = \delta, f_{\lambda,\delta})$

$$\begin{aligned} f(\lambda, \delta) \quad &= \quad c_0 + \sum_{i=0}^{1} c_{i+1}\lambda^{1-i}\delta^i + \sum_{i=0}^{2} c_{i+3}\lambda^{2-i}\delta^i + \sum_{i=0}^{3} c_{i+6}\lambda^{3-i}\delta^i + \cdots \\ &\quad + \sum_{i=0}^{p-1} c_{i+\alpha}\lambda^{p-1-i}\delta^i + \sum_{i=0}^{p} c_{i+\alpha+p}\lambda^{p-i}\delta^i. \end{aligned} \tag{3. 12}$$

The residual is defined as

$$\begin{aligned} J(c_0, c_1, \cdots, c_{2p+\alpha}) \quad &= \quad \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} w_{\lambda,\delta}\Big(f_{\lambda,\delta} - c_0 - \sum_{i=0}^{1} c_{i+1}\lambda^{1-i}\delta^i - \cdots \\ &\quad - \sum_{i=0}^{p} c_{i+\alpha+p}\lambda^{p-i}\delta^i\Big)^2. \end{aligned} \tag{3. 13}$$

After solving the normal equations, the coefficients $c_{2p+\alpha}$ can be determined. Once the values of all the $c_{2p+\alpha}$ coefficients are obtained, they can be substituted into equation (3. 12) and get the best fit polynomial.

Remark 3.1. *An alternative approach to obtain the normal equations for c_0, c_1, c_2, $\cdots$, $c_{2p-1+\alpha}$, and $c_{2p+\alpha}$ is by multiplying them with equation (3. 12).*

$$\sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} w_{\lambda,\delta},\ \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \lambda w_{\lambda,\delta},\ \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \delta w_{\lambda,\delta},\ \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \lambda^2 w_{\lambda,\delta},\ \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \lambda\delta w_{\lambda,\delta}$$
$$\sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \delta^2 w_{\lambda,\delta},\cdots,\ \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \lambda\delta^{p-1} w_{\lambda,\delta}, \quad \textit{and} \quad \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \delta^{p} w_{\lambda,\delta} \textit{respectively.}$$

Step 2: The $(2\nu+1)^2$-point b-ary surface schemes are derived by putting $(\lambda,\delta) = (\pm\frac{2t+1}{2b}, \pm\frac{2t+1}{2b})$, where $b \geq 2$ and $t = 0, 1, 2, \cdots, b-1$, into the best fit polynomial of any degree. In this derivation, we replace $f_{\lambda,\delta}$ with $f^{k}_{i+\lambda,j+\delta}$ and $f(\lambda,\delta)$ with $f^{k+1}_{bi+\alpha,bj+\alpha}$. Similarly, the even points surface schemes are obtained when $-\nu+1 \leq \lambda,\delta \leq \nu$, where $\nu \geq 1$. In this case, the weight function $w_{\lambda,\delta}$ is used for the surface schemes.

3.2. **The 1^{st} degree polynomial and $(2\nu+1)$-point, b-ary bivariate schemes.** The linear bivariate polynomial can be obtained by substituting $p = 1$ into equation (3. 12).

$$f(\lambda,\delta) = c_0 + c_1\lambda + c_2\delta. \tag{3. 14}$$

The corresponding normal equations are

$$\begin{aligned} \eta_{0,\lambda,\delta} &= c_0\xi_{0,\lambda,\delta} + c_1\xi_{1,\lambda,\delta} + c_2\xi_{2,\lambda,\delta}, \\ \eta_{1,\lambda,\delta} &= c_0\xi_{1,\lambda,\delta} + c_1\xi_{3,\lambda,\delta} + c_2\xi_{4,\lambda,\delta}, \\ \eta_{2,\lambda,\delta} &= c_0\xi_{2,\lambda,\delta} + c_1\xi_{4,\lambda,\delta} + c_2\xi_{5,\lambda,\delta}, \end{aligned}$$

where

$$\xi_{0,\lambda,\delta} = \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} w_{\lambda,\delta}, \quad \xi_{1,\lambda,\delta} = \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \lambda w_{\lambda,\delta}, \quad \xi_{2,\lambda,\delta} = \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \delta w_{\lambda,\delta},$$
$$\xi_{3,\lambda,\delta} = \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \lambda^2 w_{\lambda,\delta}, \quad \xi_{4,\lambda,\delta} = \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \delta\lambda w_{\lambda,\delta}, \quad \xi_{5,\lambda,\delta} = \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \delta^2 w_{\lambda,\delta},$$
$$\eta_{0,\lambda,\delta} = \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} f_{\lambda,\delta} w_{\lambda,\delta}, \quad \eta_{1,\lambda,\delta} = \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \lambda f_{\lambda,\delta} w_{\lambda,\delta},$$
$$\text{and} \quad \eta_{2,\lambda,\delta} = \sum_{\lambda=-\nu}^{\nu}\sum_{\delta=-\nu}^{\nu} \delta f_{\lambda,\delta} w_{\lambda,\delta}.$$

The values of c_0, c_1 and c_2 are obtained by solving the normal equations:

$$\begin{aligned} c_0 &= \frac{1}{\gamma_{\lambda,\delta}}\Big\{-\eta_{0,\lambda,\delta}\xi^2_{3,\lambda,\delta}+\xi_{3,\lambda,\delta}\xi_{1,\lambda,\delta}\eta_{2,\lambda,\delta}+\xi_{3,\lambda,\delta}\xi_{2,\lambda,\delta}\eta_{1,\lambda,\delta}-\eta_{1,\lambda,\delta}\xi_{4,\lambda,\delta}\xi_{1,\lambda,\delta} \\ &\quad +\xi_{2,\lambda,\delta}\xi_{4,\lambda,\delta}\eta_{0,\lambda,\delta}-\xi^2_{2,\lambda,\delta}\eta_{2,\lambda,\delta}\Big\}, \qquad (3.\ 15) \\ c_1 &= \frac{1}{\gamma_{\lambda,\delta}}\Big\{-\eta_{0,\lambda,\delta}\xi^2_{3,\lambda,\delta}+\xi_{3,\lambda,\delta}\xi_{1,\lambda,\delta}\eta_{2,\lambda,\delta}+\xi_{3,\lambda,\delta}\xi_{2,\lambda,\delta}\eta_{1,\lambda,\delta}-\eta_{1,\lambda,\delta}\xi_{4,\lambda,\delta}\xi_{1,\lambda,\delta} \\ &\quad +\xi_{2,\lambda,\delta}\xi_{4,\lambda,\delta}\eta_{0,\lambda,\delta}-\xi^2_{2,\lambda,\delta}\eta_{2,\lambda,\delta}\Big\}, \qquad (3.\ 16) \\ c_2 &= \frac{1}{\gamma_{\lambda,\delta}}\Big\{-\eta_{0,\lambda,\delta}\xi^2_{3,\lambda,\delta}+\xi_{3,\lambda,\delta}\xi_{1,\lambda,\delta}\eta_{2,\lambda,\delta}+\xi_{3,\lambda,\delta}\xi_{2,\lambda,\delta}\eta_{1,\lambda,\delta}-\eta_{1,\lambda,\delta}\xi_{4,\lambda,\delta}\xi_{1,\lambda,\delta} \\ &\quad +\xi_{2,\lambda,\delta}\xi_{4,\lambda,\delta}\eta_{0,\lambda,\delta}-\xi^2_{2,\lambda,\delta}\eta_{2,\lambda,\delta}\Big\}, \qquad (3.\ 17) \end{aligned}$$

and

$$\gamma_{\lambda,\delta}=\xi_{0,\lambda,\delta}\xi_{2,\lambda,\delta}\xi_{4,\lambda,\delta}-\xi_{0,\lambda,\delta}\xi^2_{3,\lambda,\delta}+2\xi_{1,\lambda,\delta}\xi_{2,\lambda,\delta}\xi_{3,\lambda,\delta}-\xi^2_{1,\lambda,\delta}\xi_{4,\lambda,\delta}-\xi^3_{2,\lambda,\delta} \qquad (3.\ 18)$$

The general form of the odd point surface schemes, derived by substituting $(\lambda,\delta)=(\pm\frac{2t+1}{2b},\pm\frac{2t+1}{2b})$ and replacing $f_{\lambda,\delta}$ with $f^k_{\lambda,\delta}$ and $f(\lambda,\delta)$ with $f^{k+1}_{bi+\alpha,bj+\alpha}$ in the best fit linear polynomial (obtained by substituting the values of c_0, c_1, and c_2 in equation (3. 12)), can be expressed as follows:

$$f^{k+1}_{bi+\alpha,bj+\alpha} = c_0+c_1\left(\pm\frac{2t+1}{2b}\right)+c_2\left(\pm\frac{2t+1}{2b}\right), \alpha=0,1\cdots,b-1 \qquad (3.\ 19)$$

where c_0, c_1, and c_2 are defined in equations (3.4), (3.5), and (3.6), respectively. For each value of ν and b, we obtain odd point b-ary non-tensor product surface schemes.

Remark 3.3. *The bivariate algorithm is capable of reproducing existing least squares-based bivariate schemes.*

- *The odd point family* [3] *reproduced after putting $w_{\lambda,\delta}=1$ and $b=2$ in (3. 19).*
- *The even point family* [3] *reproduced after putting $w_{\lambda,\delta}=1$, $b=2$ and $-\nu+1\le\lambda,\delta\le\nu$ in (3. 19).*
- *The non-stationary part of odd point schemes presented in* [3] *are obtained by replacing $w_{\lambda,\delta}$ with $w^k_{\lambda,\delta}=\cos(\frac{x-\lambda}{2^k})\cos(\frac{x-\delta}{2^k})$ at $x=\frac{1}{4}$ and $b=2$ in (3. 19).*
- *The non-stationary part of even point schemes presented in* [3] *are obtained by replacing $w_{\lambda,\delta}$ with $w^k_{\lambda,\delta}=\cos(\frac{x-\lambda}{2^k})\cos(\frac{x-\delta}{2^k})$ at $x=\frac{1}{4}$, $b=2$ and $-\nu+1\le\lambda,\delta\le\nu$ in (3. 19).*
- *The odd point family* [10] *reproduced after setting $w_{\lambda,\delta}=1$ and $p=3$, in Section 3.*
- *The odd point family* [10] *reproduced after setting $w_{\lambda,\delta}=1$, $p=3$ and $-\nu+1\le\lambda,\delta\le\nu$ in Section 3.*

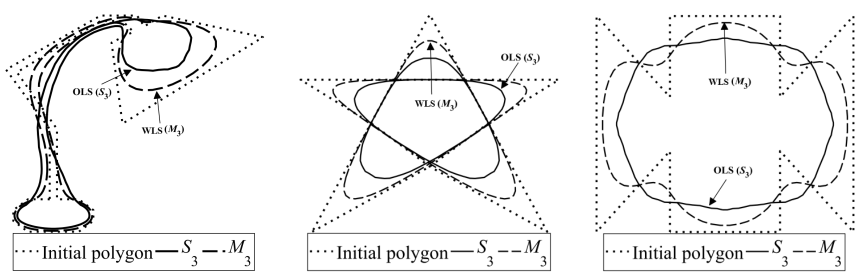
4. Applications and comparison of the schemes

In this section, we present several experiments to check the performance of the schemes. We examine the performance of the schemes by fitting noise-free and noisy data. We also consider noisy data with outliers. In addition to these experiments, we test the scheme for the generation of fundamental curve shapes. Finally, we have generated refined surfaces at various subdivision levels. These experiments collectively demonstrate the effectiveness and versatility of the schemes in various scenarios. We also highlighted the potential of the schemes for curve and surface modelling.

We initiate the comparison between the schemes generated by the WLS method and the LS method as described in [3]. Figure 1 illustrates that the scheme M_3 exhibits better curve quality and shape preservation compared to the scheme S_3 presented in [3]. In Figure 2, we observe that both WLS and LS schemes show similar visual performance in the presence of noisy data with outliers. Figure 3 showcases the visual performance of the 3-point non-stationary scheme, which is the counterpart of the scheme presented in [3]. Here, M_3^k represents the 3-point non-stationary scheme obtained from (2.10) when $k = 0$ and $k = 2$, using the WLS method. The quality of the curve produced by the scheme (2.10) is superior to the LS-based scheme when $k = 0$, although it is more sensitive to outliers. When $k = 2$, the scheme exhibits reduced sensitivity to outliers. However, the quality of the curves generated by the WLS and LS schemes, as reported by [3], is equivalent. Figure 4 demonstrates the reproduction of conics.

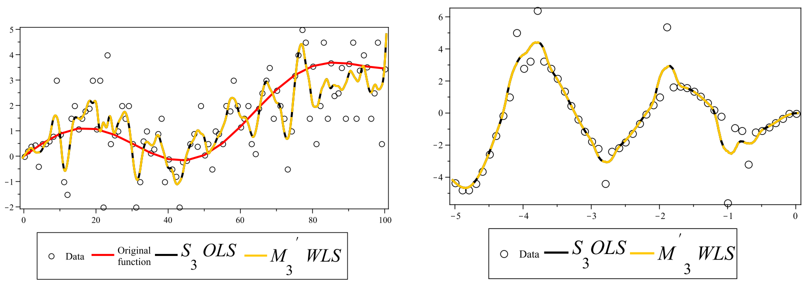
In the WLS method, we have various options for constructing classical and statistical subdivision schemes using weight functions. The subdivision schemes generated by the weight function $w_\lambda = \frac{1}{|x-\lambda|}$ exhibit classical behaviour. Therefore, the M_3 scheme produces significantly different curves compared to the S_3 subdivision scheme for non-noisy data. However, for noisy data, both schemes yield similar results.

Finally, we present the applications of the 16-point surface scheme in Figure 5 and Figure 6. These figures illustrate the application of the scheme on two different sketches or polygonal shapes. The resulting refined sketches are also depicted in the figures, presenting the fine details and smoothness achieved by the scheme.



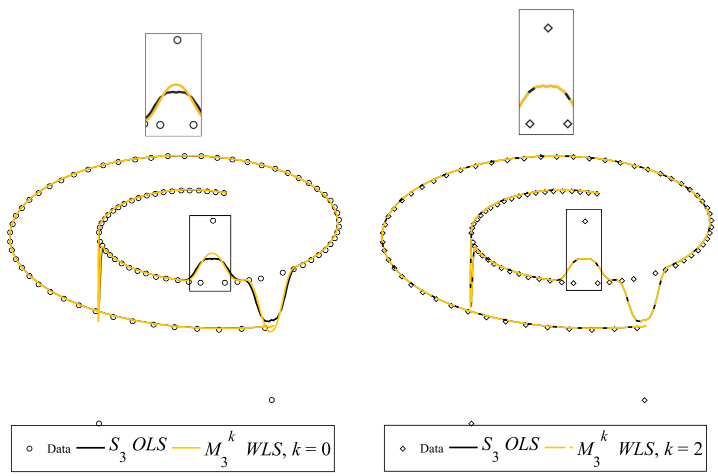
$(\mathbf{a})$ $(\mathbf{b})$ $(\mathbf{c})$

FIGURE 1. $(\mathbf{a})$, $(\mathbf{b})$ and $(\mathbf{c})$ are the comparisons of 3-point schemes of S_3 and M_3 in terms of noise free data.



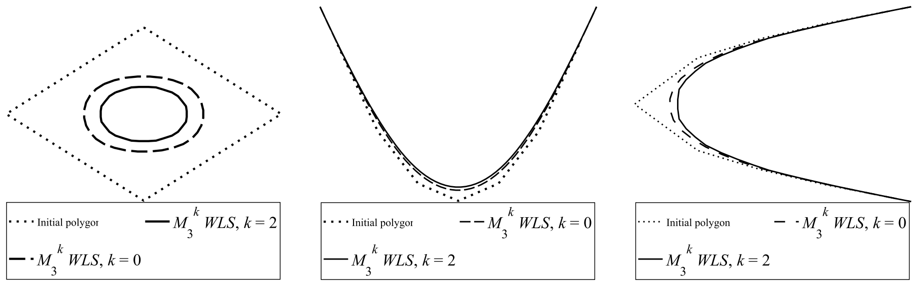
($\mathbf{a}$) ($\mathbf{b}$)

FIGURE 2. ($\mathbf{a}$) and ($\mathbf{b}$) present the comparison of 3-point schemes of S_3 and M_3 in terms of noisy data and outliers, respectively.



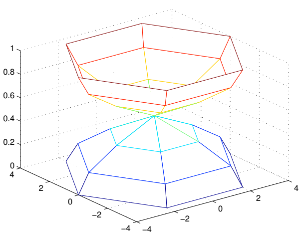
($\mathbf{a}$) ($\mathbf{b}$)

FIGURE 3. ($\mathbf{a}$) and ($\mathbf{b}$) present the comparison of 3-point scheme S_3 and non-stationary 3-point scheme M_3^k at levels $k = 0$ and $k = 2$ when data contains outliers, respectively.

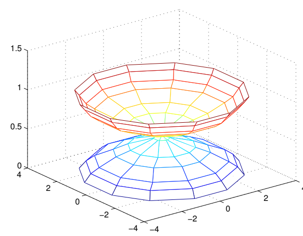


(**a**) (**b**) (**c**)

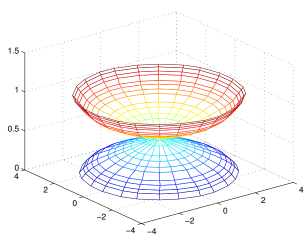
FIGURE 4. Reproduction of conics by a scheme corresponding to M_3^k.



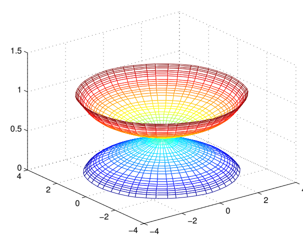
(**a**) Initial mesh



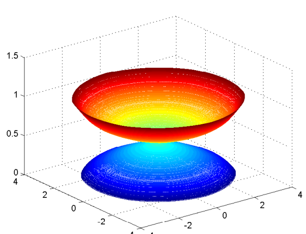
(**b**) First level



(**c**) Second level

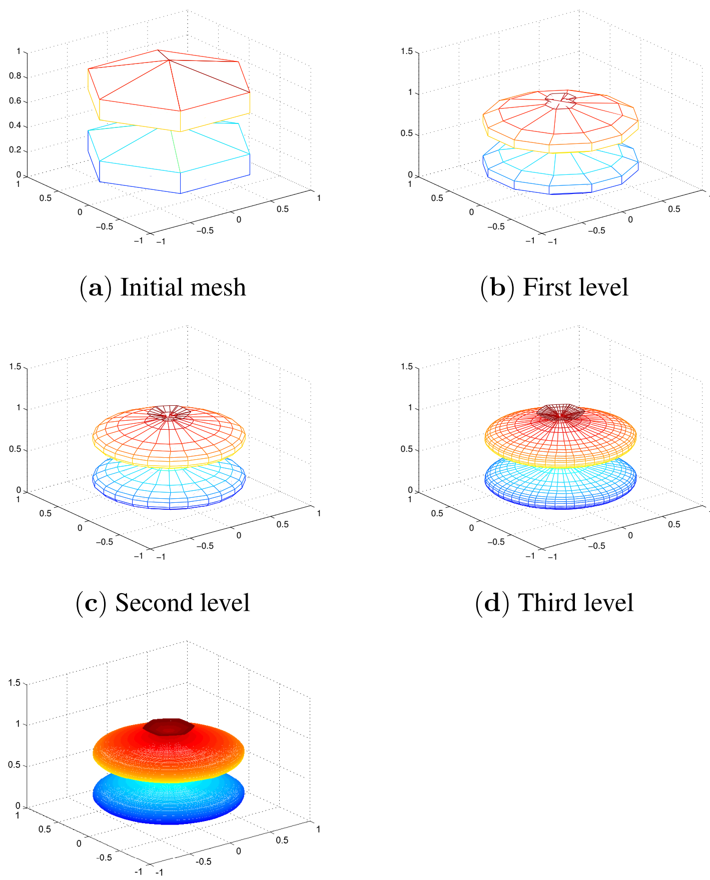


(**d**) Third level



(**e**) Limit surface

FIGURE 5. (a) Shows the initial mesh, whereas (b)-(e) are produced by the 16-point binary surface scheme.



(**a**) Initial mesh

(**b**) First level

(**c**) Second level

(**d**) Third level

(**e**) Limit surface

FIGURE 6. (a) Shows the initial mesh, whereas (b)-(e) are produced by the 16-point binary surface scheme.

5. Conclusions

In conclusion, this article explored the application of the WLS method in the development of subdivision schemes. Compared to the traditional LS approximation, the WLS method offered a more flexible approach, resulting in subdivision schemes capable of generating smooth curves and surfaces without explicit function representations. The article presented generalized algorithms based on WLS for the generation of both stationary and non-stationary subdivision schemes, providing a comprehensive framework for their implementation. Furthermore, a comparison between LS and WLS-based schemes highlighted that the schemes produced by the LS method were a special case of those generated by WLS.

To assess the performance of the proposed schemes, several experiments were conducted. Noise-free data fitting was performed to evaluate the accuracy and quality of the generated curves, demonstrating the capability of the schemes produced by WLS method. The ability of the schemes to handle outliers was assessed using noisy data with outlier points. The schemes were also tested on fundamental shapes of curves to ensure accurate capture of shape characteristics. Furthermore, the generation of surface models at various subdivision levels presented the ability of the schemes to produce detailed and refined surfaces.

Conflict of interests

The authors declare that there is no conflict of interest regarding the publication of this article and regarding the funding that they have received.

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Authors Contribution

Muhammad Asghar: Conceptualization, Methodology, Writing – Original Draft. Ghulam Mustafa: Supervision, Formal Analysis, Writing – Review & Editing. Faheem Khan: Data Curation, Validation, Writing – Review & Editing. Rakib Mustafa: Visualization, Software, Formatting. All authors read and approved the final manuscript.

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