

Degree-Based Index Optimization in Trees with Pendant Constraints

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Abstract: Graphs are used in mathematics to mathematically depict networks, which are essentially collections of interconnected things. The topology and structure of networks and molecular graphs can be better understood with the help of topological indices which are practical mathematical tools. In this article, we examine lower bounds topological indices, including the Gourva, hyper Gourava, Forgotten and hyper Forgotten indices, in a unified manner within the group of trees with η vertices and η_1 pendent vertices. Our goal is to derive sharp inequality and describe the associated extremal graphs. Lower bounds for a number of vertex-degree-based topological indices are provided by the primary findings. We make use of some graphs transformations to compute the TI for fixed pendant vertices and order n . These limitations are novel even for the Gourva, hyper Gourava, Forgotten and hyper Forgotten indices. Trees with a fixed number of pendant vertices maximize scalability, energy efficiency, and multicast communication in networks like peer-to-peer systems and Wireless Sensor Networks (WSNs). Fixed pendant vertex trees aid in the modeling of molecular structures with certain bonding properties in chemical graph theory, improving stability and medication effectiveness.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09

Key Words: Fixed parameters, Extremal Graphs, Graph transformations.

1. INTRODUCTION

The study of graphs, which are mathematical structures made up of vertices (or nodes) joined by edges, is known as graph theory. The previous few decades have seen a remarkable development in the discipline of chemical graph theory, providing us with numerous novel and distinctive concepts and instruments. Chemical graph theory offers several interesting applications, one of which is the study of vertex-degree based topological indices with a specific breadth. In most cases, studying topological indices involves determining

the extremal values of a certain topological index of a graph with particular constraints and determining the corresponding extremal graphs. A topological index is a numerical quantity that can be used to describe the qualities or activities of organic substances, and it is important in medicine, materials science, and chemistry, among other fields. They are a useful means of translating chemical compositions into numerical information that can be used for comparisons with physical attributes see. One of the most well-known degree-based topological indices in chemical graph theory is the Randić' index. Since its inception, it has been extensively studied and used to illustrate the traits that distinguish specific molecular structures. In chemical exercises, the Randić' index is commonly used, especially in QSPR/QSAR investigations. The Randić' index was proposed by Randić' [17] and is known as

$$R_\alpha(G) = \sum_{xy \in E(G)} (d_G(x)d_G(y))^\alpha$$

The General Sum-connectivity index, which Zhou and Trinajstić' [22] developed by altering the Randić index concept, has the following definition.

$$\chi_\alpha(G) = \sum_{xy \in E(G)} (d_G(x) + d_G(y))^\alpha$$

Gutman and Trinajstić' [8] gave approximated formulas for the structure-dependency of total π -electron energy more than 30 years ago.

$$M_1(G) = \sum_{xy \in E(G)} (d_G(x) + d_G(y))$$

$$M_2(G) = \sum_{xy \in E(G)} (d_G(x)d_G(y))$$

The graph invariants defined through the following function:

$$TI(G) = \sum_{xy \in E(G)} F(d_G(x)d_G(y)) = \sum_{xy \in E(G)} F(s, t)$$

Where $d_G(x) = s$ and $d_G(x) = t$. Motivated by the Zagreb indices' criteria and possible uses, V. R. Kulli [12, 13] introduced the first, second and hyper Gourava indices in the manner described below. Forgotten topological index was named and first studied by Furtula and Gutman [6] in 2015, F-index and hyper F-index are defined as

| Topological Indices (TI's) | Symbolic representation | $F(s, t)$ |
|---------------------------------|-------------------------|-------------------|
| First Gourava Index [12] | $GO_1(G)$ | $(s + t + st)$ |
| Second Gourava Index [12] | $GO_2(G)$ | $(s + t)(st)$ |
| First Hyper Gourava Index [13] | $HGO_1(G)$ | $(s + t + st)^2$ |
| Second Hyper Gourava Index [13] | $HGO_2(G)$ | $((s + t)(st))^2$ |
| Forgotten Index | $F(G)$ | $s^2 + t^2$ |
| Hyper Forgotten Index | $HF(G)$ | $(s^2 + t^2)^2$ |

Relationships, dependencies, and interactions are frequently represented using networks and graphs in the data sciences. For a variety of applications, the structural complexity and similarity of such networks are analyzed through the use of Gourava indices. The toxicity,

activity, and medical efficacy of a drug are influenced by its structure, including the addition or removal of atoms or bonds. Chemical applicability is demonstrated by the Gourava and hyper-Gourava indices [4]. In this study, only simple, connected, undirected graphs are taken into consideration. Assume that τ is a graph. Then, its vertex and edge sets are denoted by $V(\tau)$ and $E(\tau)$, respectively. Here, $d_\tau(\mu)$ denotes a vertex's degree; however, when there is no uncertainty regarding the graph in question, we use $d(\mu)$ instead of $d_\tau(\mu)$. A vertex of degree one is pendant vertex. A star graph S_{τ_η} is $\kappa_{1,\eta-1}$ on η vertices.

Recently, Maitreyi *et al.* [14] found the lowest Sombor index of trees with fixed order pendant vertices. Farooq *et al.* [5] identified trees that have varying total eccentricity indices. In [10] authors listed all molecular trees with a constant number of pendant vertices and identified the extremal trees among them in terms of the symmetric division deg index. A lower bound on the symmetric division deg index is also obtained. The task of identifying graphs with the lowest value over the class of all trees with a specified order and a fixed number of pendant vertices for specific values of α is addressed in [2]. The graphs with the lowest ABS index among all trees with a predetermined number of pendant vertices are characterized in [3]. Additionally, a partial solution to the problem of finding graphs with the least ABS index in the class of all trees with a fixed order and number of pendant vertices is given. The study in [19] fully ascertain the extreme values of the Sigma index of linked networks with fixed pendant vertices and describe the related extremal graphs. Recently S. Akram *et al.* [1] characterized extremal graphs and demonstrate the ordering of various subfamilies of graphs with respect to F-index. More specifically authors established the lower and upper bounds of concerned graphs for fixed pendant vertices. S. Naureen *et al.* [15] derived the sharp lower and upper constraints for the modified first Zagreb connection index of trees with fixed order and number of pendant vertices and characterized the related extremal trees as well. A significant research presented by Yuedan Yao [20] that ascertain the least general sum-connectivity index of all trees and chemical trees with given pendant vertices, together with the corresponding extremal tree. The graphs that extremize the Wiener index among all graphs on fixed vertices and pendant vertices are characterized in [16]. Goubko constructed sharp lower bounds for the first and second Zagreb indices (respectively)[7] with respect to trees and chemical trees with the specified number of pendant vertices and determined optimum trees. Later on an alternative proof of this relation was given in [9] by Gutman and Goubko. The study by A. Yu in [21] presented trees with maximal Merrifield-Simmons indices and minimal Hosoya indices, (respectively) for the trees with fixed pendant vertices.

Recently a lot of work has been done on networks or acyclic structures [18]. The study in [11] established the sharp extremal bounds for the Forgotten index. Furthermore it presents the theoretical approach to integrate artificial intelligence (AI) and machine learning (ML) methodologies with traditional methods. Atoms that are only joined to the molecular structure by a single bond are referred to be pendant vertices, or terminal vertices, in drug structures. Pendant vertices are frequently associated with functional groups that influence drug reactivity and interactions with biological targets. Pendant vertices serve an important, albeit frequently peripheral, function in the overall network structure, assisting data scientists in understanding the data's periphery, outliers, and edges. Pendant vertices are crucial for a drug's biological activity, solubility, reactivity, and general bodily function since their number frequently correlates to important chemical and pharmacological characteristics. A

fixed number of pendant vertices allows systems to achieve balanced design, stability, efficiency, and scalability customized to specific usage scenarios. The basic goal of studying topological indices is to determine constraints on the indices that fix various parameters. Inspired by literature presented we examine a few extreme problems on the supplied parameters *i.e.* fixed order η and pendant vertices η_1 . This study aims to develop new sharp inequalities for topological indices $TI = \{GO_1, GO_2, HGO_1, HGO_2, F, HF\}$, in a unified manner and to characterize those graphs that are extremal with respect to these indices.

2. METHODOLOGY

In order to identify extremal structures with regard to different TI , (where $TI = \{GO_1, GO_2, HGO_1, HGO_2, F, HF\}$), we examine the class of trees with fixed order η and a given number of pendant vertices η_1 . We do this by applying a number of graph transformation that maintain pendant vertices and vertex count. We build possible extremal trees and calculate the relevant indices by iteratively implementing these transformation. By doing this, we identify the trees that achieve the highest or lowest index values and obtain precise boundaries for every index in the specified class of trees.

3. RESULTS

In this section we present the results concerning the extremal values of TI , where $TI = \{GO_1, GO_2, HGO_1, HGO_2, F, HF\}$.

Transformation 1: Let τ represent a tree with two or more branching vertices. Consider a path of τ , $\mu_1\mu_2\dots\mu_\gamma$ which is pendant such that $\gamma \geq 3$ with $d(\mu_\gamma) = 1$. Take a vertex $\omega \in V(\tau)$ that is located on the exclusive route connecting μ_1 and another vertex of τ which is branching. We get $\tau' = \tau - \mu_2\mu_3 + \mu_1\omega + \mu_1\mu_3 + \mu_\gamma\omega$.

Lemma 1. Let τ and τ' represent trees in transformation 1 (see Fig.1) then $TI(\tau) > TI(\tau')$.

Proof. Take that $d(\mu_1) \geq 3$ and $d(\omega) \geq 2$. Here we know that $TI = \{GO_1, GO_2, HGO_1, HGO_2, F, HF\}$, then we have

(1)

$$\begin{aligned} GO_1(\tau) - GO_1(\tau') &= (d(\mu_1) + d(\omega) + d(\mu_1)d(\omega)) + 8 + 5 - (d(\omega) + 2 + 2d(\omega)) - (2d(\mu_1) + 1) \\ &= d(\omega) - 1)(d(\mu_1) - 2) + 8 > 0 \end{aligned}$$

$$GO_1(\tau) - GO_1(\tau') > 0$$

(2)

$$\begin{aligned} GO_2(\tau) - GO_2(\tau') &= (d(\mu_1)d(\omega))(d(\mu_1) + d(\omega)) + 16 + 6 - 2d(\omega)(2 + d(\omega)) - d(\mu_1)(d(\mu_1) + 1) \\ &= d(\mu_1^2d(\omega) - d(\mu_1)^2 + d(\omega)^2d(\mu_1) - 2d(\omega)^2 - 4d(\omega) - d(\mu_1) + 22 > 0 \end{aligned}$$

$$GO_2(\tau) - GO_2(\tau') > 0$$

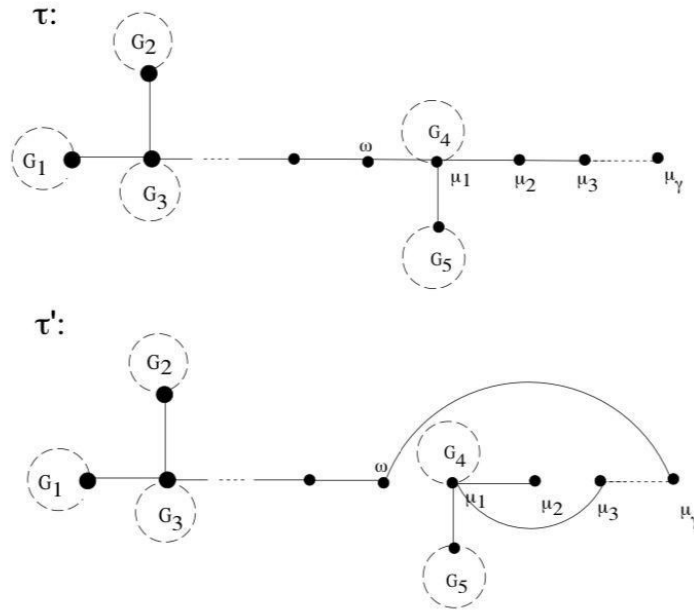


FIGURE 1. The τ and τ' are the trees considered in transformation 1 and G_i is the subtree that may or may not be trivial where $i = 1, 2, \dots, 5$.

(3)

$$\begin{aligned} HGO_1(\tau) - HGO_1(\tau') &= (d(\mu_1) + d(\omega) + d(\mu_1)d(\omega))^2 + 64 + 25 - (d(\omega) + 2 + 2d(\omega))^2 - (2d(\mu_1) + 1)^2 \\ &= -3d(\mu_1)^2 - 8d(\omega)^2 + d(\mu_1)^2d(\omega)^2 + 2d(\mu_1)d(\omega) + 2d(\mu_1)^2d(\omega) + 2d(\mu_1)d(\omega)^2 \\ &\quad - 14d(\omega) - 4d(\mu_1) + 84 > 0 \end{aligned}$$

$$HGO_1(\tau) - HGO_1(\tau') > 0$$

(4)

$$\begin{aligned} HGO_2(\tau) - HGO_2(\tau') &= (d(\mu_1)^2d(\omega)) + (d(\mu_1d(\omega)^2))^2 + 256 + 36 - (4d(\omega) + 2d(\omega)^2)^2 - (d(\mu_1)^2 + d(\mu_1)^2)^2 \\ &= (d(\omega) - 1)[d(\mu_1)^4(d(\omega) + 1 + 2d(\mu_1)^3(d(\omega)^2 + d(\omega) + 1) + d(\mu_1)^2(d(\omega) + 1)(d(\omega)^2 + 1) \\ &\quad - 4d(\omega)^2(d(\omega) + 2)^2 > 0 \end{aligned}$$

$$HGO_2(\tau) - HGO_2(\tau') > 0$$

(5)

$$\begin{aligned} F(\tau) - F(\tau') &= d(\mu_1)^2 + d(\omega)^2 + 8 + 5 - d(\omega)^2 - 4 - d(\mu_1)^2 - 1 \\ &= 8 > 0 \end{aligned}$$

$$F(\tau) - F(\tau') > 0$$

(6)

$$\begin{aligned} HF(\tau) - HF(\tau') &= (d(\mu_1)^2 + d(\omega)^2)^2 + 64 + 25 - (d(\omega)^2 + 4)^2 - (d(\mu_1)^2 + 1)^2 \\ &= (d(\mu_1) - 4)(2d(\omega)^2 - 2) + 64 > 0 \end{aligned}$$

$$HF(\tau) - HF(\tau') > 0$$

and thus the lemma is valid for TI . \square

Let $\mu_1, \mu_2, \dots, \mu_\kappa$ be an internal path of τ . Following path is non-trivial with μ_1, μ_κ as a branching vertices and $d(\mu_i) = 2$ for $2 \leq i \leq \kappa - 1$. Take a class $\mathbb{T}(\eta, \eta_1)$ consisting of trees τ where η is order and η_1 as pendant vertices with 3 as maximum degree which allow us to take each pendant path of length one and each internal path of length not less than two.

Theorem 1 Consider a tree τ of η^{th} order with η_1 as pendant vertices. Then we have

$$TI(\tau) \geq \begin{cases} 8\eta + 5\eta_1 - 26 = \varpi_{GO_1}(\eta, \eta_1), & TI = GO_1 \\ 16\eta + 24\eta_1 - 100 = \varpi_{GO_2}(\eta, \eta_1), & TI = GO_2 \\ 64\eta + 99\eta_1 - 406 = \varpi_{HGO_1}(\eta, \eta_1), & TI = HGO_1 \\ 256\eta + 1176\eta_1 - 4120 = \varpi_{HGO_2}(\eta, \eta_1), & TI = HGO_2 \\ 8\eta + 12\eta_1 - 38 = \varpi_F(\eta, \eta_1), & TI = F \\ 64\eta + 246\eta_1 - 694 = \varpi_{HF}(\eta, \eta_1), & TI = HF \end{cases}$$

and this equality prevails if and only if $\tau \in \mathbb{T}(\eta, \eta_1)$.

Proof. The outcome can be easily be confirmed for $\eta = 1, 2, 3, 4, 5$. Assume next that $\eta \geq 6$ for $\eta_1 = \eta - 1$ we possess $\tau = S\tau_\eta$, thus

(1)

$$GO_1(S\tau_\eta) - \varpi_{GO_1}(\eta, \eta_1) = 2\eta^2 - 16\eta + 32 > 0$$

(2)

$$GO_2(S\tau_\eta) - \varpi_{GO_2}(\eta, \eta_1) = \eta^3 - 2\eta^2 - 39\eta + 124 > 0$$

(3)

$$HGO_1(S\tau_\eta) - \varpi_{HGO_1}(\eta, \eta_1) = 4\eta^3 - 8\eta^2 - 158\eta + 504 > 0$$

(4)

$$HGO_2(S\tau_\eta) - \varpi_{HGO_2}(\eta, \eta_1) = \eta^5 - 3\eta^4 + 3\eta^3 - \eta^2 - 1432\eta + 5296 > 0$$

(5)

$$F(S\tau_\eta) - \varpi_F(\eta, \eta_1) = \eta^3 - 3\eta^2 - 16\eta + 48 > 0$$

(6)

$$HF(S\tau_\eta) - \varpi_{HF}(\eta, \eta_1) = \eta^5 - 5\eta^4 + 12\eta^3 - 16\eta^2 - 298\eta + 936 > 0$$

Hereinafter we consider $2 \leq \eta_1 \leq \eta - 2$ and $\eta \geq 6$. By executing mathematical induction on η_1 we get the outcome, for $\eta_1 = 2$ we get $\tau = \rho_\eta$.

(1)

$$GO_1(\rho_\eta) - \varpi_{GO_1}(\eta, 2) = 8\eta - 14 - 8\eta - 10 + 26 = 2 > 0$$

(2)

$$GO_2(\rho_\eta) - \varpi_{GO_2}(\eta, 2) = 16\eta - 36 - 16\eta - 48 + 100 = 16 > 0$$

(3)

$$HGO_1(\rho_\eta) - \varpi_{HGO_1}(\eta, 2) = 64\eta - 142 - 64\eta - 198 + 406 = 66 > 0$$

(4)

$$HGO_2(\rho_\eta) - \varpi_{HGO_2}(\eta, 2) = 256\eta - 696 - 256\eta - 2352 + 4120 = 1072 > 0$$

(5)

$$F(\rho_\eta) - \varpi_F(\eta, 2) = 8\eta - 14 - 8\eta - 24 + 38 = 0$$

(6)

$$HF(\rho_\eta) - \varpi_{HF}(\eta, 2) = 64\eta - 142 - 64\eta - 492 + 694 = 60 > 0$$

Further we consider $\eta_1 \geq 3$ and the theorem is satisfied for $\eta_1 - 1$ such that $\eta_1 \leq \eta - 2$ and $\eta \geq 6$. Two cases arise here. The theorem is organized in the following way:

Case 1: There are no pendent paths in tree τ that are longer than 1.

There are 4 subcases of case 1.

Subcase 1: In τ there must be one pendent vertex with neighboring vertex having degree greater than 3.

Subcase 2: In τ each pendent vertex is next to a degree 3 vertex, and at least one vertex of degree 3 vertex has just one pendent neighbor.

Subcase 3: There must be a vertex of τ with degree 3 whose one neighbor is branching and two are pendent and each pendent vertex is next to a vertex with degree 3.

Subcase 4: Each pendent vertex must be next to a degree 3 vertex whose one neighbor is of degree 2 and two are pendants.

Case 2: There is at least one pendent path having length longer than one, in τ .

There are two subcases of case 2.

Subcase 1: τ has a single branching vertex.

Subcase 2: τ has a minimum of two branching vertices

Case 1. There are no pendent paths in tree τ that are longer than 1.

Subcase 1. In τ there must be one pendent vertex with neighboring vertex having degree greater than 3.

Consider $\mu \in V(\tau)$ and $d(\mu) = l$ with m pendant neighbors which are $\mu_0, \mu_1, \dots, \mu_{m-1}$ as well as $l - m$ neighbors which are non pendant $\mu_m, \mu_{m+1}, \dots, \mu_{l-1}$. Since $l \geq 4$, $m \geq 1$, $\eta \geq 6$ thus we have $l - m > 0$. By considering these mentioned inequalities and applying induction hypothesis we get.

(1)

$$\begin{aligned}
GO_1(\tau) - GO_1(\tau - \mu_0) &= 2m + 2l - 1 + \sum_{k=m}^{l-1} [1 + d(\mu_k)] \\
&\geq 2m + 2l - 1 + (l - m)3 \\
&\geq GO_1(\tau - \mu_0) + 5l - m - 1 \\
&\geq \varpi_{GO_1}(\eta, \eta_1) + 5l - m - 14 > \varpi_{GO_1}(\eta, \eta_1)
\end{aligned}$$

(2)

$$\begin{aligned}
GO_2(\tau) - GO_2(\tau - \mu_0) &= 2lm + l^2 - s + \sum_{k=m}^{l-1} [d(\mu_k)^2 + d(\mu_k)(2l - 1)] \\
&\geq 2lm + l^2 - s + (l - m)(4l + 2) \\
&\geq GO_2(\tau - \mu_0) + 5l^2 - 2lm + l - m \\
&\geq \varpi_{GO_2}(\eta, \eta_1) + 5l^2 - 2lm + l - m - 40 > \varpi_{GO_2}(\eta, \eta_1)
\end{aligned}$$

(3)

$$\begin{aligned}
HGO_1(\tau) - HGO_1(\tau - \mu_0) &= 4l^2 + 8lm - 4l + 1 + \sum_{k=m}^{l-1} [d(\mu_k)^2(2l + 1) + 4ld(\mu_k) + 2l - 1] \\
&\geq 4l^2 + 8lm - 4l + 1 + (l - m)(18l + 3) \\
&\geq HGO_1(\tau - \mu_0) + 22l^2 - 10lm - l - 3m + 1 \\
&\geq \varpi_{HGO_1}(\eta, \eta_1) + 22l^2 - 10lm - l - 3m - 162 > \varpi_{HGO_1}(\eta, \eta_1)
\end{aligned}$$

(4)

$$\begin{aligned}
HGO_2(\tau) - HGO_2(\tau - \mu_0) &= l^4 - 2l^3 + l^2 + 4ml^2 \\
&\quad + \sum_{k=m}^{l-1} [d(\mu_k)^4(2l - 1) + 2d(\mu_k)^3(2l^2 - 3l + 2) + d(\mu_k)^2(4l^3 - 6l^2 + 4l - 1)] \\
&\geq l^4 - 2l^3 + l^2 + 4ml^2 + (l - m)4(4l^3 + 2l^2 + 3) \\
&\geq HGO_2(\tau - \mu_0) + 5l^4 + 10l^3 + 9l^2 + 12 \\
&\geq \varpi_{HGO_2}(\eta, \eta_1) + 5l^4 + 10l^3 + 9l^2 - 1490 > \varpi_{HGO_2}(\eta, \eta_1)
\end{aligned}$$

(5)

$$\begin{aligned}
F(\tau) - F(\tau - \mu_0) &= ms^2 + m + \sum_{k=m}^{l-1} [2l - 1] - (m - 1)(l^2 - 2l + 2) \\
&\geq F(\tau - \mu_0) + 3l^2 - 3l + 2 \\
&\geq \varpi_F(\eta, \eta_1) + 3l^2 - 3l - 19 > \varpi_F(\eta, \eta_1)
\end{aligned}$$

(6)

$$\begin{aligned}
HF(\tau) - HF(\tau - \mu_0) &= l^4 - 4l^3 + 8l^2 + 4ml^3 - 6ml^2 - 3m - 8l + 4 \\
&\quad + \sum_{k=m}^{l-1} [2(l - 1)d(\mu_k)^2 + 2l^3 - 6l^2 + 4l - 1] \\
&\geq l^4 - 4l^3 + 8l^2 + 4ml^3 - 6ml^2 - 3m - 8l + 4 + (l - m)(8(l - 1) + 2l^3 - 6l^2 + 4l - 1) \\
&\geq HF(\tau - \mu_0) + 3l^4 + l^3(2m - 10) + 20l^2 + l(-4m - 17) + 16m + 4 \\
&\geq \varpi_{HF}(\eta, \eta_1) + 3l^4 + 2ml^3 + 16l^2 - 10l^3 + 4l(l - m) - 11l - 6(l - m) > \varpi_{HF}(\eta, \eta_1)
\end{aligned}$$

Subcase 2. In τ each pendent vertex is next to a degree 3 vertex, and at least one vertex of degree 3 vertex has just one pendent neighbor.

Take $\mu \in V(\tau)$ where $d(\mu) = 3$ and $\mu_0 \in N_\tau(\mu)$ which is pendant. Take $\mu_1, \mu_2 \in N_\tau(\mu)$ as non-pendant vertices. Using theory of induction, we obtain.

(1)

$$\begin{aligned} GO_1(\tau) - GO_1(\tau - \mu_0) &= 7 + \sum_{x=1}^2 [1 + d(\mu_x)] \\ &\geq GO_1(\tau - \mu_0) + 7 + 2(1 + 2) \\ &\geq \varpi_{GO_1}(\eta, \eta_1) \end{aligned}$$

$GO_1(\tau) = \varpi_{GO_1}(\eta, \eta_1)$ if and only if $d(\mu_1) = d(\mu_2) = 2$ with $\tau - \mu_0 \in \mathbb{T}(\eta - 1, \eta_1 - 1)$.

(2)

$$\begin{aligned} GO_2(\tau) - GO_2(\tau - \mu_0) &= 12 + \sum_x^2 [d(\mu_x)^2 + 5d(\mu_x)] \\ &\geq GO_2(\tau - \mu_0) + 2(4 + 5(2)) \\ &\geq GO_2(\tau - \mu_0) + 5l^2 - 2lm + l - m \\ &\geq \varpi_{GO_2}(\eta, \eta_1) \end{aligned}$$

$GO_2(\tau) = \varpi_{GO_2}(\eta, \eta_1)$ if and only if $d(\mu_1) = d(\mu_2) = 2$ with $\tau - \mu_0 \in \mathbb{T}(\eta - 1, \eta_1 - 1)$.

(3)

$$\begin{aligned} HGO_1(\tau) - HGO_1(\tau - \mu_0) &= 49 + \sum_{x=1}^2 [7d(\mu_x)^2 + 12d(\mu_k) + 5] \\ &\geq HGO_1(\tau - \mu_0) + 49 + 2(57) \\ &\geq \varpi_{HGO_1}(\eta, \eta_1) \end{aligned}$$

$HGO_1(\tau) = \varpi_{HGO_1}(\eta, \eta_1)$ if and only if $d(\mu_1) = d(\mu_2) = 2$ with $\tau - \mu_0 \in \mathbb{T}(\eta - 1, \eta_1 - 1)$.

(4)

$$\begin{aligned} HGO_2(\tau) - HGO_2(\tau - \mu_0) &= 144 + \sum_{x=1}^2 [5d(\mu_x)^4 + 65d(\mu_k)^3n] \\ &\geq HGO_2(\tau - \mu_0) + 144 + 2(80 + 260 + 304) \\ &\geq \varpi_{HGO_2}(\eta, \eta_1) \end{aligned}$$

$HGO_2(\tau) = \varpi_{HGO_2}(\eta, \eta_1)$ if and only if $d(\mu_1) = d(\mu_2) = 2$ with $\tau - \mu_0 \in \mathbb{T}(\eta - 1, \eta_1 - 1)$.

(5)

$$\begin{aligned} F(\tau) - F(\tau - \mu_0) &= 10 + \sum_{x=1}^2 [5] \\ &\geq F(\tau - \mu_0) + 20 \\ &\geq \varpi_F(\eta, \eta_1) \end{aligned}$$

$F(\tau) = \varpi_F(\eta, \eta_1)$ if and only if $d(\mu_1) = d(\mu_2) = 2$ with $\tau - \mu_0 \in \mathbb{T}(\eta - 1, \eta_1 - 1)$.

(6)

$$\begin{aligned} HF(\tau) - HF(\tau - \mu_0) &= 100 + \sum_{x=1}^2 [10d(\mu_k)^2 + 65] \\ &\geq HF(\tau - \mu_0) + 100 + 2(105) \\ &\geq \varpi_{HF}(\eta, \eta_1) \end{aligned}$$

$HF(\tau) = \varpi_{HF}(\eta, \eta_1)$ if and only if $d(\mu_1) = d(\mu_2) = 2$ with $\tau - \mu_0 \in \mathbb{T}(\eta - 1, \eta_1 - 1)$.

Subcase 3. There must be a vertex of τ with degree 3 whose one neighbor is branching and two are pendent and each pendent vertex is next to a vertex with degree 3.

Suppose that $d(\mu) = 3$ with two pendent neighbors μ_0 and μ_1 and μ_2 be the non dependent neighbor. Applying the principle of induction, we arrive at.

(1)

$$\begin{aligned} GO_1(\tau) - GO_1(\tau - \{\mu_0, \mu_1\}) &= 2d(\mu_2) + 16 \\ &\geq GO_1(\tau - \{\mu_0, \mu_1\}) + 22 \\ &\geq \varpi_{GO_1}(\eta, \eta_1) + 1 > \varpi_{GO_1}(\eta, \eta_1) \end{aligned}$$

(2)

$$\begin{aligned} GO_2(\tau) - GO_2(\tau - \{\mu_0, \mu_1\}) &= 2d(\mu_2)^2 + 8d(\mu_2) + 24 \\ &\geq GO_2(\tau - \mu_0, \mu_1) + 66 \\ &\geq \varpi_{GO_2}(\eta, \eta_1) + 10 > \varpi_{GO_2}(\eta, \eta_1) \end{aligned}$$

(3)

$$\begin{aligned} HGO_1(\tau) - HGO_1(\tau - \{\mu_0, \mu_1\}) &= 16d(\mu_2)^2 + 20d(\mu_2) + 106 \\ &\geq \varpi_{HGO_1}(\eta, \eta_1) + 47 > \varpi_{HGO_1}(\eta, \eta_1) \end{aligned}$$

(4)

$$\begin{aligned} HGO_2(\tau) - HGO_2(\tau - \{\mu_0, \mu_1\}) &= 8d(\mu_2)^4 + 52d(\mu_2)^3 + 80d(\mu_2)^2 + 288 \\ &\geq \varpi_{HGO_2}(\eta, \eta_1) + 1372 > \varpi_{HGO_2}(\eta, \eta_1) \end{aligned}$$

(5)

$$\begin{aligned} F(\tau) - F(\tau - \{\mu_0, \mu_1\}) &= 28 \\ &\geq F(\tau - \{\mu_0, \mu_1\}) + 28 \\ &= \varpi_F(\eta, \eta_1) \end{aligned}$$

(6)

$$\begin{aligned} HF(\tau) - HF(\tau - \{\mu_0, \mu_1\}) &= 16d(\mu_2)^2 + 280 \\ &\geq HF(\tau - \{\mu_0, \mu_1\}) + 424 \\ &\geq \varpi_{HF}(\eta, \eta_1) + 50 > \varpi_{HF}(\eta, \eta_1) \end{aligned}$$

Subcase 4. Each pendent vertex must be next to a degree 3 vertex whose one neighbor is of degree 2 and two are pendants.

If $\tau \in (\eta, \eta_1)$ with maximum degree three and there is no combination of neighboring vertices with degree three in τ then we obtain $TI(\tau) = \varpi_{TI}(\eta, \eta_1)$.

Let maximum degree of τ is 3 with $d(\mu) = d(\nu) = 3$ where $\mu\nu \in V(\tau)$. Assume that $\omega_1 \in V(\tau)$, $d(\omega_1) = 3$ with two pendent vertices and one vertex ω of degree 2 and $N(\omega) = \{\omega_1, \omega_2\}$. Consider $\tau' = \tau - \mu\nu - \omega_1\omega - \omega_2\omega + \omega_1\omega_2 + \mu\omega + \nu\omega$. Moreover τ and τ' have similar degree pattern. As maximum degree is 3 of τ so it is valid that $d(\omega_2) = 2$ or 3. Conversely, nevertheless, we have

(1)

$$GO_1(\tau) - GO_1(\tau') = -d(\omega_2) + 3 \geq 0$$

(2)

$$GO_2(\tau) - GO_2(\tau') = -d(\omega_2)^2 - 5d(\omega_2) + 24 \geq 0$$

(3)

$$HGO_1(\tau) - HGO_1(\tau') = -d(\omega_2)^2 - 12d(\omega_2) + 99 \geq 0$$

(4)

$$HGO_2(\tau) - HGO_2(\tau') = -5d(\omega_2)^4 - 38d(\omega_2)^3 - 65d(\omega_2)^2 + 2016 \geq 0$$

(5)

$$F(\tau) - F(\tau') \geq 0$$

(6)

$$HF(\tau) - HF(\tau') = -10d(\omega_2)^2 + 90 \geq 0$$

Recognize that if $d(\omega_2) = 3$ then $TI(\tau) = TI(\tau')$ and τ' has a vertex say ω_1 with $d(\omega_1) = 3$ with two dependent neighbors and one non-pendant with degree 3 neighbor. Hence It can be inferred from Subcase 3 that $TI(\tau) = TI(\tau') > \varpi_{TI}(\eta, \eta_1)$.

$TI(\tau) - TI(\tau') > 0$ for $d(\omega_2) = 2$. The process is accomplished if $\tau \in \mathbb{T}(\eta, \eta_1)$. If τ' has at least one pair of neighboring branching vertices, we carry out the previous procedure once more until the intended outcome is reached.

In the scenario under consideration, where the maximum degree of τ is more than 3, the intended outcome still needs to be proven. Given a vertex μ' of maximum degree Δ and Δ is at least 4 and there is another vertex ν' with $d(\nu') = 3$ where $\mu', \nu' \in V(\tau)$. Assume that ν' has two pendent neighbors and a neighbor of degree 2 and $N_\tau(\mu') = \{\mu_1, \mu_2, \dots, \mu_\Delta\}$. Say that ν_1 and ν' are neighbors and ν_1 is pendant. We assume that μ_2 is on the exclusive $\mu' - \nu'$ path without affecting generality. Take $\tau'' = \tau - \mu_1\mu' + \mu_1\nu'$. The tree τ'' has η vertices, of which $\eta_1 - 1$ are pendent, this is evident to observe. We have arrived at this conclusion by applying the inductive hypothesis and the conditions $d_\tau(\mu) = \Delta$ and $\Delta \geq 4$, $d_\tau(\mu_\iota) \geq 2$ where $\iota = \{1, 2, \dots, \Delta\}$, we get

(1)

$$\begin{aligned}
GO_1(\tau) - GO_1(\tau'') &= \sum_{i=2}^{\Delta} [d(\mu_i) + 1] + \Delta + \Delta d(\mu_1) - 2d(\mu_1) - 6 \\
GO_1(\tau) &\geq GO_1(\tau'') + (\Delta - 1)(2 + 1) + \Delta - 2 + d\mu_1(\Delta - 2) - 4 \\
GO_1(\tau) &\geq \varpi_{GO_1}(\eta, \eta_1) + 6 > \varpi_{GO_1}(\eta, \eta_1)
\end{aligned}$$

(2)

$$\begin{aligned}
GO_2(\tau) - GO_2(\tau'') &= \sum_{i=2}^{\Delta} [d(\mu_i)^2 + 2\Delta d(\mu_i) - d(\mu_i)] + \Delta d(\mu_1)^2 + \Delta^2 d(\mu_1)^2 - 2d(\mu_1)^2 - 4d(\mu_1) - 18 \\
GO_2(\tau) &\geq GO_2(\tau'') + (\Delta - 1)[4 + 4\Delta - 2] + d(\mu_1)^2(\Delta - 2) + d(\mu_1)(\Delta^2 - 4) - 18 \\
&\geq \varpi_{GO_2}(\eta, \eta_1) + 18 > \varpi_{GO_2}(\eta, \eta_1)
\end{aligned}$$

(3)

$$\begin{aligned}
HGO_1(\tau) - HGO_1(\tau'') &= \sum_{i=2}^{\Delta} [(d(\mu_i) + \Delta + \Delta d(\mu_i))^2 - (\Delta - 1 + \Delta d(\mu_i))^2] + (d\mu_1 + \Delta + \Delta d\mu_1)^2 \\
&\quad - (3d\mu_1 + 2)^2 + (7)^2 - (11)^2 \\
HGO_1(\tau) &\geq HGO_1(\tau'') + (\Delta - 1)(3 + 18\Delta) + (1 + d\mu_1)[\Delta^2(1 + d\mu_1) + 2\Delta d\mu_1 - 8d\mu_1 - 4] - 72 \\
HGO_1(\tau) &\geq \varpi_{HGO_1}(\eta, \eta_1) + 210 > \varpi_{HGO_1}(\eta, \eta_1)
\end{aligned}$$

(4)

$$\begin{aligned}
HGO_2(\tau) - HGO_2(\tau'') &= \sum_{i=2}^{\Delta} [(\Delta d(\mu_i)^2 + \Delta^2 d(\mu_i))^2 - (\Delta d(\mu_i)^2 + \Delta^2 d(\mu_i) - 2\gamma d(\mu_i) - d(\mu_i)^2 + d(\mu_i))^2] \\
&\quad + (\Delta d\mu_1^2 + \Delta^2 d\mu_1)^2 - (2d\mu_1^2 + 4d\mu_1)^2 + (12)^2 - (30)^2 \\
HGO_2(\tau) &\geq HGO_2(\tau'') + (\Delta - 1)[16\Delta^3 + 24\Delta^2 - 4] + \Delta^4 + 16\Delta^3 + 16\Delta^2 + (12)^2 - (30)^2 \\
HGO_2(\tau) &\geq \varpi_{HGO_2}(\eta, \eta_1) + 20\Delta^4 + 24\Delta^3 - 8\Delta^2 - 4\Delta - 2184 > \varpi_{HGO_2}(\eta, \eta_1)
\end{aligned}$$

(5)

$$\begin{aligned}
F(\tau) - F(\tau'') &= \sum_{i=2}^{\Delta} [2\Delta - 1] + \Delta^2 - 7 \\
F(\tau) &\geq F(\tau'') + (\Delta - 1)(2\Delta - 1) + \Delta^2 - 7 \\
F(\tau) &\geq \varpi_F(\eta, \eta_1) + 18 > \varpi_F(\eta, \eta_1)
\end{aligned}$$

(6)

$$\begin{aligned}
HF(\tau) - HF(\tau'') &= \sum_{i=2}^{\Delta} [\Delta^2 + d(\mu_i)^2 - (\Delta^2 + d(\mu_i)^2 - 2\Delta + 1)^2] + (\Delta^2 + d\mu_1^2)^2 - (d\mu_1^2 + 4)^2 + (10)^2 - (13)^2 \\
HF(\tau) &\geq HF(\tau'') + (\Delta - 1) [4\Delta^3 - 6\Delta^2 + 20\Delta - 9] + \Delta^4 + 8\Delta^2 - 32 - 85 \\
HF(\tau) &\geq \varpi_{HF}(\eta, \eta_1) + 5\Delta^4 - 10\Delta^3 + 34\Delta^2 - 29\Delta - 354 \\
&\quad \varpi_{HF}(\eta, \eta_1) + 714 > \varpi_{HF}(\eta, \eta_1)
\end{aligned}$$

Case 2. There is at least one pendent path having length longer than one, in τ .**Subcase 1.** τ has a single branching vertex.

Suppose that τ has a minimum of two pendent paths with a minimum length of two, given that $\mu_0\mu_1\mu_2\dots\mu_\gamma$ and $\nu_0\nu_1\nu_2\dots\nu_\delta$ be such two pendent paths with μ_0 and ν_0 as branching vertex and μ_γ, ν_δ as pendants. Assume that $\tau' = \tau - \mu_1\mu_2 + \mu_2\nu_\delta$ with η_1 is at least 3. We get for $TI = \{GO_1, GO_2, HGO_1, HGO_2, F, HF\}$

(1)

$$\begin{aligned}
GO_1(\tau) - GO_1(\tau') &= (3\eta_1 + 2) - (2\eta_1 + 1) - 3 \\
&\geq \eta_1 - 2 > 0
\end{aligned}$$

(2)

$$\begin{aligned}
GO_2(\tau) - GO_2(\tau') &= (2\eta_1^2 + 4\eta_1) + (\eta_1^2 + \eta_1 - 10) \\
&\geq \eta_1^2 + 3\eta_1 - 10 > 0
\end{aligned}$$

(3)

$$\begin{aligned}
HGO_1(\tau) - HGO_1(\tau') &= (3\eta_1 + 2)^2 - (2\eta_1 + 1)^2 - 8^2 + 5^2 \\
&\geq 5\eta_1 + 2)^2 + 8\eta_1 + 2) - 36 > 0
\end{aligned}$$

(4)

$$\begin{aligned}
HGO_2(\tau) - HGO_2(\tau') &= (2\eta_1^2 + 4\eta_1)^2 + (\eta_1^2 + \eta_1)^2 - 16^2 + 6^2 \\
&\geq 3\eta_1^4 + \eta_1^3 + 15\eta_1^2 - 220 > 0
\end{aligned}$$

(5)

$$F(\tau) - F(\tau') \geq 0$$

(6)

$$\begin{aligned}
HF(\tau) - HF(\tau') &= (\eta_1^2 + 4)^2 - (\eta_1^2 + 1)^2 - 8^2 + 5^2 \\
&\geq 6(\eta_1 - 2)(\eta_1 + 2) > 0
\end{aligned}$$

After considering the aforementioned explanation, we determine that if τ has a single branching vertex, then

(1)

$$GO_1(\tau) \geq GO_1(\tau^*) = \tau(\eta, \eta_1) = (\eta_1 - 1)(2\eta_1 + 1)(3\eta_1 + 2) + 8\eta - 8\eta_1 - 11$$

(2)

$$GO_2(\tau) \geq GO_2(\tau^*) = (\eta_1 - 1)(\eta_1^2 + \eta_1) + (2\eta_1^2 + 4\eta_1) + (\eta - \eta_1 - 2)16 + 6$$

(3)

$$HGO_1(\tau) \geq HGO_1(\tau^*) = (\eta_1 - 1)(2\eta_1 + 1)^2 + (3\eta_1 + 2)^2 + (\eta - \eta_1 - 2)(8)^2 + (5)^2$$

(4)

$$HGO_2(\tau) \geq HGO_2(\tau^*) = (\eta_1 - 1)(\eta_1^2 + \eta_1)^2 + (2\eta_1^2 + 4\eta_1)^2 + (\eta - \eta_1 - 2)(16)^2 + (6)^2$$

(5)

$$F(\tau) \geq F(\tau^*) = (\eta_1 - 1)(\eta_1^2 + 1) + (\eta_1^2 + 4) + (\eta - \eta_1 - 2)(8) + 5$$

(6)

$$HF(\tau) \geq HF(\tau^*) = (\eta_1 - 1)(\eta_1^2 + 1)^2 + (\eta_1^2 + 4)^2 + (\eta - \eta_1 - 2)(8)^2 + (5)^2$$

In this case, τ^* is the tree created by joining one of the pendent vertices of the path graph $\rho_{\eta-\eta_1+1}$ to $\eta_1 - 1$ pendent vertices. It is confirmed by simple computations that the inequality holds for $\eta_1 \geq 3$.

(1)

$$(\eta_1 - 1)(2\eta_1 + 1)(3\eta_1 + 2) + 8\eta - 8\eta_1 - 11 > \varpi_{GO_1}(\eta, \eta_1)$$

$$2\eta_1^2 - 6\eta_1 - 10 > 8\eta + 5\eta_1 - 26$$

$$2\eta_1^2 - 11\eta_1 + 16 > 0$$

Thus, we get

$$GO_1(\tau) \geq GO_1(\tau^*) > \varpi_{GO_1}(\eta, \eta_1)$$

(2)

$$(\eta_1 - 1)(\eta_1^2 + \eta_1) + (2\eta_1^2 + 4\eta_1) + (\eta - \eta_1 - 2)16 + 6 > \varpi_{GO_2}(\eta, \eta_1)$$

$$\eta_1^3 + 2\eta_1^2 - 13\eta_1 + 16\eta - 26 > 16\eta + 24\eta_1 - 100$$

$$\eta_1^3 + 2\eta_1^2 - 37\eta_1 + 74 > 0$$

Thus, we get

$$GO_2(\tau) \geq GO_2(\tau^*) > \varpi_{GO_2}(\eta, \eta_1)$$

(3)

$$(\eta_1 - 1)(2\eta_1 + 1)^2 + (3\eta_1 + 2)^2 + (\eta - \eta_1 - 2)(8)^2 + (5)^2 \geq \varpi_{HGO_1}(\eta, \eta_1)$$

Thus, we get

$$HGO_1(\tau) \geq HGO_1(\tau^*) > \varpi_{HGO_1}(\eta, \eta_1)$$

(4)

$$(\eta_1 - 1)(\eta_1^2 + \eta_1)^2 + (2\eta_1^2 + 4\eta_1)^2 + (\eta - \eta_1 - 2)(16)^2 + (6)^2 > \varpi_{HGO_2}(\eta, \eta_1)$$

$$\eta_1^5 + 5\eta_1^4 + 15\eta_1^3 + 15\eta_1^2 - 256\eta_1 + 256\eta - 476 > 256\eta + 1176\eta_1 - 4120$$

$$\eta_1^5 + 5\eta_1^4 + 15\eta_1^3 + 15\eta_1^2 - 1432\eta_1 + 3644 > 0$$

Thus, we get

$$HGO_2(\tau) \geq HGO_2(\tau^*) > \varpi_{HGO_2}(\eta, \eta_1)$$

(5)

$$\begin{aligned}
(\eta_1 - 1)(\eta_1^2 + 1) + (\eta_1^2 + 4) + (\eta - \eta_1 - 2)(8) + 5 &> \varpi_F(\eta, \eta_1) \\
\eta_1^3 - 7\eta_1 + 8\eta - 12 &> 8\eta + 12\eta_1 - 38 \\
\eta_1^3 - 19\eta_1 - 26 &> 0
\end{aligned}$$

Thus, we get

$$F(\tau) \geq F(\tau^*) > \varpi_F(\eta, \eta_1)$$

(6)

$$\begin{aligned}
(\eta_1 - 1)(\eta_1^2 + 1)^2 + (\eta_1^2 + 4)^2 + (\eta - \eta_1 - 2)(8)^2 + (5)^2 &> \varpi_{HF}(\eta, \eta_1) \\
\eta_1^5 + 2\eta_1^3 + 6\eta_1^2 - 63\eta_1 + 64\eta - 88 &> 64\eta + 246\eta_1 - 694 \\
\eta_1^5 + 2\eta_1^3 + 6\eta_1^2 - 297 &> 0
\end{aligned}$$

Thus, we get

$$HF(\tau) \geq HF(\tau^*) > \varpi_{HF}(\eta, \eta_1)$$

Subcase 2. τ has a minimum of two branching vertices.

There exists a tree τ^* of rank η with η_1 pendent vertices, by Lemma 1, such that $TI(\tau) > TI(\tau^*)$ and τ^* has no pendent path of length larger than 1. Nevertheless, according to Case 1, $TI(\tau^*) \geq TI(\varpi_{TI}(\eta, \eta_1))$. Thus, $TI(\tau) > TI(\varpi_{TI}(\eta, \eta_1))$. This concludes the evidence. \square

EXAMPLE

Consider the trees $\{T, T_1, T_2, T_3\}$ for $\eta = 12$ and $\eta_1 = 4$ and $\{T', T'_1, T'_2, T'_3\}$ for $\eta = 12$ and $\eta_1 = 5$ as shown in Figure 2 and Figure 3 respectively. By direct computations we can have their $TI = \{GO_1, HGO_1, GO_2, HGO_2, F, HF\}$ and numerical values are given in Table 1 and Table 2. It is clearly seen that $TI(T) \leq TI(T_j)$ and similarly $TI(T') \leq TI(T'_j)$ where $j = 1, 2, 3, 4, 5$ which satisfies our theorem.

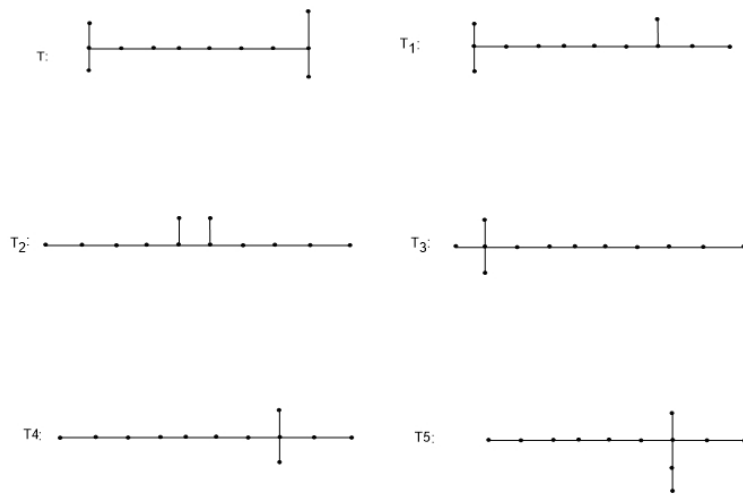
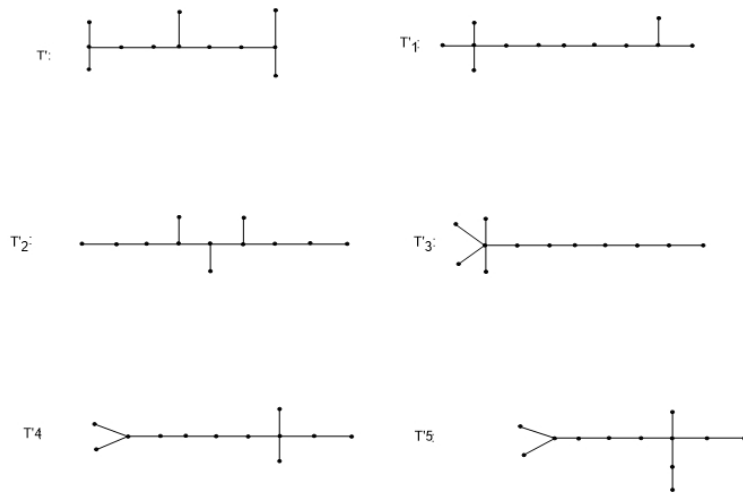
FIGURE 2. Trees with $\eta = 12$ and $\eta_1 = 4$ FIGURE 3. Trees with $\eta = 12$ and $\eta_1 = 5$

TABLE 1. $\eta = 12$ and $\eta_1 = 4$

| TI | T | T_1 | T_2 | T_3 | T_4 | T_5 |
|---------|------|-------|-------|-------|-------|-------|
| GO_1 | 90 | 91 | 93 | 93 | 96 | 98 |
| HGO_1 | 758 | 791 | 871 | 848 | 924 | 1000 |
| GO_2 | 188 | 196 | 214 | 210 | 212 | 246 |
| HGO_2 | 3656 | 4192 | 5164 | 5076 | 6104 | 8444 |
| F | 106 | 106 | 106 | 124 | 124 | 140 |
| HF | 1058 | 1088 | 1168 | 1548 | 1748 | 1820 |

TABLE 2. $\eta = 12$ and $\eta_1 = 5$

| TI | T' | T'_1 | T'_2 | T'_3 | T'_4 | T'_5 |
|---------|------|--------|--------|--------|--------|--------|
| GO_1 | 95 | 106 | 98 | 99 | 100 | 102 |
| HGO_1 | 857 | 914 | 1017 | 1113 | 990 | 1066 |
| GO_2 | 212 | 226 | 248 | 276 | 244 | 250 |
| HGO_2 | 4832 | 5716 | 8648 | 9816 | 7400 | 9084 |
| HF | 118 | 136 | 118 | 178 | 136 | 136 |
| HF | 1304 | 2460 | 1464 | 3890 | 1964 | 2036 |

4. CONCLUSION

It became evident that a psychological perspective alone could not account for the complexity of the events occurring in the classroom. Establishing social norms that provided the setting in which children engaged in meaningful activity was an aspect of social interaction not considered prior to the classroom teaching experiment. As these norms became accepted, the students participated in a type of discourse in which they were expected to explain and justify their solutions and listen to others. The teacher acted to initiate and guide students' learning by posing questions and highlighting children's expectations. As students engaged in this discourse, their personal meanings were negotiated until an agreement was reached. The establishment of taken-as-shared meanings between the participants enabled mathematical ideas to be established by members of the class.

AUTHOR CONTRIBUTIONS

Conceptualization, Salma Kanwal; methodology, software, Khadija Sarwar ; validation, Salma Kanwal and Khadija Sarwar; investigation, Salma Kanwal; writing-original draft, Khadija Sarwar; writing-review and editing, Salma Kanwal and Khadija Sarwar; visualization, Salma Kanwal and Khadija Sarwar; funding acquisition, Salma Kanwal and Khadija Sarwar. All authors have read and agreed to the published version of the manuscript.

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CONFLICTS OF INTEREST

Authors declared no conflicts of interest.

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